The Curse of Dimensionality for Numerical Integration of Smooth Functions

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Abstract

We prove the curse of dimensionality for multivariate integration of $C^k$ functions. The proofs are based on volume estimates for $k = 1$ together with smoothing by convolution. This allows us to obtain smooth fooling functions for $k > 1$.

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1 Introduction

We study multivariate integration for different classes $F_d$ of smooth functions $f : \mathbb{R}^d \to \mathbb{R}$. Our emphasis is on large values of $d \in \mathbb{N}$. We want to approximate

$$S_d(f) = \int_{D_d} f(x) \, dx \quad \text{for} \quad f \in F_d \quad (1)$$

up to some error $\varepsilon > 0$, where $D_d \subset \mathbb{R}^d$ has (Lebesgue) measure 1. The results in this paper hold for arbitrary sets $D_d$, the standard example of course is $D_d = [0,1]^d$.

We consider (deterministic) algorithms that use only function values. We consider classes $F_d$ of functions bounded in absolute value by 1 and containing all constant functions $f(x) \equiv c$ with $|c| \leq 1$. This implies that the initial error is one, i.e.,

$$\inf_{c \in \mathbb{R}} \max_{f \in F_d} |S_d(f) - c| = \max_{f \in F_d} |S_d(f)| = 1,$$

so that multivariate integration is well scaled and that is why we consider $\varepsilon < 1$.

Let $n(\varepsilon, F_d)$ denote the minimal number of function values needed for this task in the worst case setting\(^1\). By the curse of dimensionality we mean that $n(\varepsilon, F_d)$ is exponentially large in $d$. That is, there are positive numbers $c$, $\varepsilon_0$ and $\gamma$ such that

$$n(\varepsilon, F_d) \geq c (1 + \gamma)^d \quad \text{for all} \quad \varepsilon \leq \varepsilon_0 \quad \text{and infinitely many} \quad d \in \mathbb{N}. \quad (2)$$

For many natural classes $F_d$ the bound in (2) will hold for all $d \in \mathbb{N}$. This applies in particular to the classes considered in this paper.

\(^1\)We add that $n(\varepsilon, F_d)$ is the information complexity of multivariate integration over $F_d$ and is proportional to the (total) complexity as long as $F_d$ is convex and symmetric. The last two assumptions are needed to guarantee that a linear algorithm is optimal and its implementation cost is linear in $n(\varepsilon, F_d)$. 

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There are many classes $F_d$ for which the curse of dimensionality has been proved, see [5, 7] for such examples. However, it has been not known if the curse of dimensionality occurs for probably the most natural class which is the unit ball of $r$ times continuously differentiable functions,

$$\mathcal{C}_d^r = \{ f \in C^r(\mathbb{R}^d) \mid \|D^\beta f\| \leq 1 \quad \text{for all} \quad |eta| \leq r \},$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$, with non-negative integers $\beta_j$, $|\beta| = \sum_{j=1}^{d} \beta_j$, and $D^\beta$ denotes the operator of $\beta_j$ times differentiation with respect to the $j$th variable for $j = 1, 2, \ldots, d$. By $\| \cdot \|$ we mean the sup norm, $\|D^\beta f\| = \sup_{x \in \mathbb{R}^d} |(D^\beta f)(x)|$.

For $r = 0$, we obviously have $n(\varepsilon, \mathcal{C}_d^0) = \infty$ for all $\varepsilon < 1$ and all $d \in \mathbb{N}$. Therefore from now on we always assume that $r \geq 1$. For $r = 1$, the curse of dimensionality for $\mathcal{C}_d^1$ follows from the results of Sukharev [8]. Whether the curse holds for $r \geq 2$ has been an open problem for many years.

The class $\mathcal{C}_d^r$ for $D_d = [0, 1]^d$ (and functions and norms restricted to $D_d$) was already studied in 1959 by Bakhvalov [2], see also [4]. He proved that there are two positive numbers $a_{d,r}$ and $A_{d,r}$ such that

$$a_{d,r} \varepsilon^{-d/r} \leq n(\varepsilon, \mathcal{C}_d^r) \leq A_{d,r} \varepsilon^{-d/r} \quad \text{for all} \quad d \in \mathbb{N} \quad \text{and} \quad \varepsilon \in (0, 1). \quad (3)$$

This means that for a fixed $d$ and for $\varepsilon$ tending to zero, we know that $n(\varepsilon, \mathcal{C}_d^r)$ is of order $\varepsilon^{-d/r}$ and the exponent of $\varepsilon^{-1}$ grows linearly\(^2\) in $d$. Unfortunately, since the known dependence on $d$ in $a_{d,r}$ is exponentially small and the known dependence on $d$ in $A_{d,r}$ is exponentially large in $d$, Bakhvalov’s result does not allow us to conclude whether the curse of dimensionality holds for the class $\mathcal{C}_d^r$. In fact, if we reverse the roles of $d$ and $\varepsilon$, and consider a fixed $\varepsilon$ and $d$ tending to infinity, the bound (3) on $n(\varepsilon, \mathcal{C}_d^r)$ is useless. We prove the following result.

**Main Theorem.** The curse of dimensionality holds for the classes $\mathcal{C}_d^r$ with the super-exponential lower bound

$$n(\varepsilon, \mathcal{C}_d^r) \geq c_r (1 - \varepsilon) d^{d/(2r+3)} \quad \text{for all} \quad d \in \mathbb{N} \quad \text{and} \quad \varepsilon \in (0, 1),$$

where $c_r \in (0, 1]$ depends only on $r$.

We also prove that the curse of dimensionality holds for even smaller classes of functions $F_d$ for which the norms of arbitrary directional derivatives are bounded proportionally to $1/\sqrt{d}$.

\(^2\)In the language of tractability, this result means that we do not have polynomial tractability but does not allow us to conclude the lack of weak tractability.
We now discuss how we obtain lower bounds on $n(\varepsilon, F_d)$ for numerical integration defined on convex and symmetric classes $F_d$. The standard proof technique is to find a fooling function $f \in F_d$ that vanishes at the points $P = \{x_1, x_2, \ldots, x_n\}$ at which we sample functions from $F_d$, and the integral of $f$ is as large as possible. All algorithms that use function values at $x_j$'s must give the same approximation of the integral of $f$ and of the integral of $-f$. That is why the integral of $f$ is a lower bound on the worst case error of all algorithms using function values at $x_j$'s. If for all choices of $x_1, x_2, \ldots, x_n$ the integral of $f$ is larger than $\varepsilon$ then we know that $n(\varepsilon, F_d) \geq n$.

We start with the fooling function

$$f_0(x) = \min \left\{ 1, \frac{1}{\delta \sqrt{d}} \text{dist}(x, P_{\delta}) \right\} \quad \text{for all} \quad x \in \mathbb{R}^d,$$

where

$$P_{\delta} = \bigcup_{i=1}^{n} B_{\delta}(x_i)$$

and $B_{\delta}(x_i)$ is the ball with center $x_i$ and radius $\delta \sqrt{d}$. The function $f_0$ is Lipschitz. By a suitable smoothing via convolution we construct a fooling function $f_r \in C_r$ and $f_r|_P = 0$.

## 2 Preliminaries

In this section we precisely define our problem. Let $F_d$ be a class of Lebesgue integrable functions $f: \mathbb{R}^d \to \mathbb{R}$. For $f \in F_d$, we approximate the integral $S_d(f)$, see (1), by algorithms

$$A_{n,d}(f) = \phi_{n,d}(f(x_1), f(x_2), \ldots, f(x_n)),$$

where $x_j \in \mathbb{R}^d$ can be chosen adaptively and $\phi_{n,d}: \mathbb{R}^n \to \mathbb{R}$ is an arbitrary mapping. Adaptation means that the selection of $x_j$ may depend on the already computed values $f(x_1), f(x_2), \ldots, f(x_{j-1})$. The (worst case) error of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}) = \sup_{f \in F_d} |S_d(f) - A_{n,d}(f)|.$$

The minimal number of function values to guarantee that the error is at most $\varepsilon$ is defined as

$$n(\varepsilon, F_d) = \min\{ n \in \mathbb{N} \mid \exists A_{n,d} \text{ such that } e(A_{n,d}) \leq \varepsilon \}.$$

Hence we minimize $n$ over all choices of adaptive sample points $x_j$ and mappings $\phi_{n,d}$. It is well known that as long as the class $F_d$ is convex and symmetric we may restrict the
minimization of $n$ by considering only nonadaptive choices of $x_j$ and linear mappings $\phi_{n,d}$. Furthermore,

$$n(\varepsilon, F_d) = \min \left\{ n \in \mathbb{N} \mid \inf_{\mathcal{P} \subset \mathbb{R}^d, \#\mathcal{P} = n} \sup_{f \in F_d, f|_{\mathcal{P}} \equiv 0} |S_d(f)| \leq \varepsilon \right\}. \quad (4)$$

see [4, Prop. 1.2.6] or [9, Theorem 5.5.1]. In this paper we always consider convex and symmetric $F_d$ so that we can use the last formula for $n(\varepsilon, F_d)$. For more details see, e.g., Chapter 4 in [5].

Observe that we allow $x_j \in \mathbb{R}^d$ instead of only $x_j \in D_d$. In this paper we are interested in lower bounds and this assumption makes our results even stronger.

As already mentioned, our lower bounds are based on a volume estimate of a neighborhood of certain sets in $\mathbb{R}^d$, see also [3]. In the following we denote by $A_\delta$, for $A \subset \mathbb{R}^d$, the $(\delta \sqrt{d})$-neighborhood of $A$, which is defined by

$$A_\delta = \{ x \in \mathbb{R}^d \mid \text{dist}(x, A) \leq \delta \sqrt{d} \}, \quad (5)$$

where $\text{dist}(x, A) = \inf_{a \in A} \|x - a\|_2$ denotes the Euclidean distance of $x$ from $A$.

Since we need the $\sqrt{d}$-scaling of the distance, we will omit it in the notation as we already did for $A_\delta$. Furthermore, we denote by $B_d^\delta(x)$ the $d$-dimensional ball with center $x \in \mathbb{R}^d$ and radius $\delta \sqrt{d}$, i.e.,

$$B_d^\delta(x) = \{ y \in \mathbb{R}^d \mid \|x - y\|_2 \leq \delta \sqrt{d} \}. \quad (5)$$

We will need some standard volume estimates for Euclidean balls. Recall that the volume of a Euclidean ball of radius 1 is given by

$$V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.$$  

From Stirling’s formula for the $\Gamma$ function we have

$$\Gamma(x + 1) = \sqrt{2\pi x} x^x e^{-x + \Theta_x \frac{1}{12x}} \quad \text{for all} \quad x > 0,$$

where $\Theta_x \in (0, 1)$, see [1, p. 257]. This leads to the estimate

$$\Gamma(x + 1) > \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \quad \text{for all} \quad x > 0.$$

Combining this estimate with the volume formula for the ball, we obtain for all $d \in \mathbb{N}$,

$$\lambda_d(B_d^\delta(x)) = (\delta \sqrt{d})^d V_d < (\delta \sqrt{d})^d \left(\frac{2\pi e}{d}\right)^{d/2} = \left(\frac{\delta \sqrt{2\pi e}}{\sqrt{\pi d}}\right)^d \left(\frac{\delta \sqrt{2\pi e}}{\sqrt{\pi d}}\right)^d < \left(\delta \sqrt{2\pi e}\right)^d. \quad (6)$$

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The volume formula for the Euclidean unit ball also shows the recurrence relation
\[ \frac{V_{d-1}}{V_d} = \frac{d}{d-1} \frac{V_{d-3}}{V_{d-2}} \quad \text{for all } d \geq 4. \]

This easily implies
\[ \frac{2}{\sqrt{d}} \frac{V_{d-1}}{V_d} < \frac{2}{\sqrt{d-2}} \frac{V_{d-3}}{V_{d-2}} \quad \text{for all } d \geq 4. \]

The last inequality can be used in an inductive argument leading to
\[ \frac{2}{\sqrt{d}} \frac{V_{d-1}}{V_d} \leq 1 \quad \text{for all } d \geq 2. \quad (7) \]

This will be needed later.

3 Convolution

In this section we fix \( k \in \mathbb{N} \) and study the convolution
\[ f_k := f \ast g_1 \ast \ldots \ast g_k \]
of a function \( f \) defined on \( \mathbb{R}^d \) with (normalized) indicator functions \( g_j \). We are interested in properties of \( f_k \) in terms of the properties of the initial function \( f \). Recall that the convolution of two functions \( f \) and \( g \) on \( \mathbb{R}^d \) is defined by
\[ (f \ast g)(x) = \int_{\mathbb{R}^d} f(x-t) g(t) \, dt \quad \text{for all } x \in \mathbb{R}^d. \]

Fix a number \( \delta > 0 \) and a sequence \( (\alpha_j)_{j=1}^k \) with \( \alpha_j > 0 \) such that
\[ \sum_{j=1}^k \alpha_j \leq 1. \]

For example, we may take \( \alpha_j = 1/k \) for \( j = 1, 2, \ldots, k \). For \( j = 1, \ldots, k \), we define the ball
\[ B_j = \left\{ x \in \mathbb{R}^d \mid \|x\|_2 \leq \alpha_j \delta \sqrt{d} \right\} \]
and the function \( g_j : \mathbb{R}^d \to \mathbb{R} \) by
\[ g_j(x) = \begin{cases} 1_{B_j}(x) & \frac{1}{\lambda_d(B_j)} = \frac{1}{\lambda_d(B_j)} \left\{ \begin{array}{ll} 1 & \text{if } x \in B_j; \\ 0 & \text{otherwise.} \end{array} \right. \end{cases} \quad (8) \]
Thus, the convolution of a function $f$ with $g_j$ can be written as
\[
(f * g_j)(x) = \frac{1}{\lambda_d(B_j)} \int_{B_j} f(x + t) \, dt \quad \text{for all} \quad x \in \mathbb{R}^d.
\]

We will frequently use the following probabilistic interpretation. Let $Y_j$ be a random variable that is uniformly distributed on $B_j$. Then the convolution of $f$ with $g_j$ can be written as the expected value
\[
(f * g_j)(x) = \mathbb{E}[f(x + Y_j)].
\]

The next theorem is the basis for the induction steps of the proofs of our main results.

For $f : \mathbb{R}^d \to \mathbb{R}$, we use the Lipschitz constant
\[
\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_2}.
\]

Define
\[
C^r = \{ f : \mathbb{R}^d \to \mathbb{R} \mid D^{\theta_1} \ldots D^{\theta_r} f \text{ is continuous for all } \ell \leq r \text{ and all } \theta_1, \ldots, \theta_r \in S^{d-1} \},
\]
where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $D^{\theta_1} f(x) = \lim_{h \to 0} \frac{1}{h} (f(x + h\theta_1) - f(x))$ is the derivative in the direction of $\theta_1$.

**Theorem 1.** For $k \in \mathbb{N}$ and $f \in C^r$, define
\[
f_k = f * g_1 * \ldots * g_k \quad \text{with } g_k \text{ from (8)}.
\]

For $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ and let $\Omega_\delta$ be its neighborhood defined as in (5). Then

(i) if $f(x) = 0$ for all $x \in \Omega_\delta$ then $f_k(x) = 0$ for all $x \in \Omega$,
(ii) $\text{Lip}(f_k) \leq \text{Lip}(f)$,
(iii) if $\int_{D_d} f(x + t) \, dx \geq \varepsilon$ for all $t \in \mathbb{R}^d$ with $\|t\|_2 \leq \delta \sqrt{d}$ then $\int_{D_d} f_k(x) \, dx \geq \varepsilon$,
(iv) for all $\ell \leq r$ and all $\theta_1, \theta_2, \ldots, \theta_r \in S^{d-1}$,
\[
\text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f_k) \leq \text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f),
\]
(v) $f_k \in C^{r+k}$, and for all $\ell \leq r$, all $j = 1, \ldots, k$ and all $\theta_1, \theta_2, \ldots, \theta_{\ell+j} \in S^{d-1}$,
\[
\text{Lip}(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \ldots D^{\theta_1} f_k) \leq \left( \prod_{i=1}^{j} \frac{1}{\delta_{\theta_{k+1-i}}} \right) \text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f).
\]
The parts (i)–(iv) of this theorem show that some properties of the initial function $f$ are preserved by convolutions. Part (v) states that we gain one “degree of smoothness” with every convolution, loosing only a multiplicative constant for its Lipschitz constant.

Proof. First note that we can write $f_k$ as

$$f_k(x) = \mathbb{E}[f(x + Y)], \quad \text{for all } x \in \mathbb{R}^d,$$

where $Y$ is a random variable with probability density function $g_1 \ast \ldots \ast g_k$. By construction of $g_k$’s which are the indicator functions of the balls whose sum of the radii is at most $\delta \sqrt{d}$, we have

$$\{t \in \mathbb{R}^d \mid g_1 \ast \ldots \ast g_k(t) > 0\} \subset \{t \in \mathbb{R}^d \mid \|t\|_2 \leq \delta \sqrt{d}\},$$

which implies that $x + Y \in \Omega_\delta$ almost surely for every $x \in \Omega$. Thus, $f(x) = 0$ for all $x \in \Omega_\delta$ implies that $f_k(x) = 0$ for all $x \in \Omega$, which is property (i).

Property (ii) is proven by

$$|f_k(x) - f_k(y)| = |\mathbb{E}[f_k(x + Y) - f_k(y + Y)]| \leq \mathbb{E} [|f(x + Y) - f(y + Y)|]$$

$$\leq \text{Lip}(f) \mathbb{E} [|f(x + Y) - (y + Y)|_2] = \text{Lip}(f) \|x - y\|_2.$$

To prove (iii), we use Fubini’s theorem and we obtain

$$\int_{D_a} f_k(x) \, dx = \int_{D_a} \mathbb{E}[f(x + Y)] \, dx = \mathbb{E} \left[ \int_{D_a} f(x + Y) \, dx \right] \geq \varepsilon$$

by assumption.

For the proof of properties (iv) and (v), let $\theta = (\theta_1, \ldots, \theta_r) \in (S^{d-1})^\ell$. We write $D^\theta$ for $D^\theta_e \ldots D^\theta_1$. Clearly, $f \in C^r$ and $\ell \leq r$ implies that $D^\theta f \in C^{r-\ell} \subseteq C$. Since $f_k$ is at least as smooth as $f$, both $D^\theta f$ and $D^\theta f_k$ are well defined.

We need the well-known fact that $D^\theta (f * g) = (D^\theta f) * g$ if $f \in C^\ell$ and $g$ has compact support. For $g = g_1 \ast \ldots \ast g_k$, we have

$$|D^\theta f_k(x) - D^\theta f_k(y)| = |((D^\theta f) * g)(x) - ((D^\theta f) * g)(y)|$$

$$= \left| \int_{\mathbb{R}^d} [(D^\theta f(x + t) - D^\theta f(y + t)] g(t) \, dt \right|$$

$$\leq \text{Lip}(D^\theta f) \|x - y\|_2 \int_{\mathbb{R}^d} g(t) \, dt$$

$$= \text{Lip}(D^\theta f) \|x - y\|_2$$

for all $x, y \in \mathbb{R}^d$. The last equality follows since the $g_k$ is normalized. This proves (iii).
For \((v)\), we need to prove that \(f_k \in C^{r+k}\) with \(f_0 = f \in C^r\) by assumption, and then it is enough to show that for all \(m \leq r + k\) and all \(\theta = (\theta_m, \ldots, \theta_1) \in (S^{d-1})^m\),

\[
\text{Lip}\left(D^\theta f_k\right) \leq \frac{1}{\delta \alpha_k} \text{Lip}\left(D^\theta f_{k-1}\right),
\]

where \(\bar{\theta} = (\theta_{m-1}, \ldots, \theta_1) \in (S^{d-1})^{m-1}\).

Assume inductively that \(f_{k-1} \in C^{m-1}\), which holds for \(k = 1\). This implies \(D^\bar{\theta}(f_{k-1} \ast g_k) = (D^\theta f_{k-1}) \ast g_k\), and

\[
D^\theta f_k(x) = D^\theta_m\left((D^\theta f_{k-1}) \ast g_k\right)(x)
\]

\[
= D^\theta_m\left(\frac{1}{\lambda_d(B_k)} \int_{\mathbb{R}^d} D^\bar{\theta} f_{k-1}(x + t) \mathbb{1}_{B_k}(t) \, dt\right)
\]

\[
= \frac{1}{\lambda_d(B_k)} D^\theta_m\left(\int_{\mathbb{R}^d} D^\bar{\theta} f_{k-1}(x + s + h\theta_m) \mathbb{1}_{B_k}(s + h\theta_m) \, dh \, ds\right)
\]

\[
= \frac{1}{\lambda_d(B_k)} \int_{\theta_m^\perp} D^\theta_m\left(\int_{\mathbb{R}^d} D^\bar{\theta} f_{k-1}(x + s + h\theta_m) \mathbb{1}_{B_k}(s + h\theta_m) \, dh \, ds\right) \, ds,
\]

where \(\theta_m^\perp\) is the hyperplane orthogonal to \(\theta_m\). For any function \(f\) on \(\mathbb{R}\) of the form

\[
f(x) = \int_{x-a}^{x+a} g(y) \, dy
\]

with some continuous function \(g\) we have

\[
f'(x) = g(x + a) - g(x - a).
\]

Therefore, we obtain

\[
D^\theta f_k(x) = \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m^\perp} D^\theta f_{k-1}\left(x + s + h_{\max}(s) \theta_m\right)
\]

\[
- D^\theta f_{k-1}\left(x + s - h_{\max}(s) \theta_m\right) \, ds
\]

with

\[
h_{\max}(s) = \max\{h \geq 0 \mid s + h\theta_m \in B_k\}.
\]

For each \(s \in B_k \cap \theta_m^\perp\), define the points \(s_1 = s + h_{\max}(s) \theta_m \in B_k\) and...
\[ s_2 = s - h_{\text{max}}(s) \theta_m \in B_k. \] Then

\[
|D^\theta f_k(x) - D^\theta f_k(y)| \leq \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m^\perp} \left| D^\theta f_{k-1}(x + s_1) - D^\theta f_{k-1}(x + s_2) - D^\theta f_{k-1}(y + s_1) + D^\theta f_{k-1}(y + s_2) \right| ds
\]

\[
\leq \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m^\perp} \left| D^\theta f_{k-1}(x + s_1) - D^\theta f_{k-1}(y + s_1) + D^\theta f_{k-1}(x + s_2) - D^\theta f_{k-1}(y + s_2) \right| ds
\]

\[
\leq \frac{2 \lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} \lambda_d(B_k) \geq \lambda_d \int_{B_k \cap \theta_m^\perp} \left| D^\theta f_{k-1}(x) - D^\theta f_{k-1}(y) \right| ds
\]

\[
\leq \frac{2 \lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} \lambda_d(B_k) \geq \lambda_d \int_{B_k \cap \theta_m^\perp} \left| D^\theta f_{k-1}(x) - D^\theta f_{k-1}(y) \right| ds
\]

In particular, this shows the implication

\[
f_{k-1} \in C^{m-1} \implies f_k \in C^m
\]

for all \( k \in \mathbb{N} \). Taking \( m = r + k \) we have \( f_k \in C^{r+k} \), as claimed.

For \( m \leq r + k \), it remains to bound \( 2 \lambda_{d-1}(B_k \cap \theta_m^\perp)/\lambda_d(B_k) \). Recall that \( B_k \) is a ball with radius \( \delta \alpha_k \sqrt{d} \) and that \( V_d \) is the volume of the Euclidean unit ball in \( \mathbb{R}^d \). We obtain from (7) that

\[
\frac{2 \lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} \geq \frac{2(\delta \alpha_k \sqrt{d})^{d-1} V_{d-1}}{(\delta \alpha_k \sqrt{d})^d V_d} = \frac{2}{\delta \alpha_k \sqrt{d}} \frac{V_{d-1}}{V_d} \leq \frac{1}{\delta \alpha_k}.
\]

\[ \square \]

4 Main Results

Let \( \mathcal{P} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d \) be a collection of \( n \) points. As pointed out in the introduction, we want to construct functions that vanish at \( \mathcal{P} \) and have a large integral. For this, we choose

\[
f_0(x) = \min \left\{ 1, \frac{1}{\delta \sqrt{d}} \text{dist}(x, \mathcal{P}_\delta) \right\}
\]

for all \( x \in \mathbb{R}^d \), where

\[
\mathcal{P}_\delta = \bigcup_{i=1}^n B_\delta^d(x_i)
\]

and \( B_\delta^d(x_i) \) is the ball with center \( x_i \) and radius \( \delta \sqrt{d} \).
The function $\text{dist}(\cdot, \mathcal{P}_\delta)$ is Lipschitz with constant 1. Hence for $\delta \leq 1$

$$\text{Lip}(f_0) = \frac{1}{\delta \sqrt{d}}. \quad (9)$$

Additionally, $f_0(x) = 0$ for all $x \in \mathcal{P}_\delta$ by definition.

Using these facts we can apply Theorem 1 to prove the curse of dimensionality for the following class of functions that are defined on $\mathbb{R}^d$. For a fixed $r \in \mathbb{N}$, we now take $\alpha_1 = \cdots = \alpha_r = \frac{1}{r}$ and define

$$F_{d,r,\delta} = \{ f : \mathbb{R}^d \to \mathbb{R} \mid f \in C^r \text{ satisfies } (10)-(12) \},$$

where

$$\|f\| \leq 1, \quad (10)$$

$$\text{Lip}(f) \leq \frac{1}{\delta \sqrt{d}}, \quad (11)$$

$$\forall k \leq r : \max_{\theta_1, \ldots, \theta_k \in S^{d-1}} \text{Lip}(D^{\theta_1} \ldots D^{\theta_k} f) \leq \frac{1}{\delta \sqrt{d}} \left( \frac{r}{\delta} \right)^k. \quad (12)$$

**Theorem 2.** For any $r \in \mathbb{N}$ and $\delta \in (0, 1]$,

$$n(\varepsilon, F_{d,r,\delta}) \geq (1 - \varepsilon) \begin{cases} 1 & \text{for } d = 1, \\ (\frac{1}{\delta \sqrt{18 e \pi}})^{-d} & \text{for } d \geq 2, \end{cases} \text{ for all } \varepsilon \in (0, 1).$$

Hence the curse of dimensionality holds for the class $F_{d,r,\delta}$ for $\delta < 1/\sqrt{18 e \pi}$.

Note that this result shows that the growth rate of $n(\varepsilon, F_{d,r,\delta})$ in $d$ can be arbitrarily large if we choose $\delta$ small enough.

**Proof.** Since the initial error for the classes $F_{d,r,\delta}$ is 1 we obtain $n(\varepsilon, F_{d,r,\delta}) \geq 1$ for all $\varepsilon \in (0, 1)$. This proves the statement for $d = 1$.

For $d \geq 2$, we use Theorem 1 with $k = r$, $\Omega = \mathcal{P}$ and $f_r(x) = f_0 * g_1 * \ldots * g_r(x)$. Here $g_k$’s are as in Theorem 1. Recall that we have chosen $\alpha_1 = \cdots = \alpha_r = 1/r$ and $\alpha_j = 0$ for $j > r$. The properties of the initial function $f_0$ and Theorem 1 immediately imply that $f_r$ satisfies (10)–(12). It remains to bound its integral. Note that $f_0(x) = 1$ for all $x \notin \mathcal{P}_{2\delta}$. Clearly, $f_r(x) \geq 0$ for all $x \in \mathbb{R}^d$. Since $f_r(x)$ depends only on the values $f_0(x+t)$ for $t \in \mathbb{R}^d$
with $\|t\|_2 \leq \delta \sqrt{d}$, it follows that $f_r(x) = 1$ for $x \notin P_3$. We thus obtain

$$
\int_{D_d} f_r(x) \, dx \geq \int_{D_d \setminus P_3} f_r(x) \, dx = 1 - \lambda_d(P_3 \cap D_d) \\
\geq 1 - \lambda_d(P_3) \geq 1 - n \lambda_d(B_{3\delta}^d) \\
> 1 - \frac{n \left(3\delta \sqrt{2e\pi}\right)^d}{\sqrt{\pi d}} \\
> 1 - n \left(3\delta \sqrt{2e\pi}\right)^d,
$$

where the last inequality follows from the bound that is given in (6). Hence $\int_{D_d} f_r(x) \, dx \leq \varepsilon$ implies that

$$
n \geq (1 - \varepsilon) (\delta \sqrt{18\varepsilon \pi})^{-d}.
$$

Since this holds for arbitrary $P$, the result follows. \hfill \Box

By Theorem 2, we know how the parameter $\delta$ comes into play. For $p > 0$, let

$$
\delta = \frac{1}{\sqrt{18\varepsilon \pi}} d^{-p/(r+1)}.
$$

For this $\delta$, we obtain a somehow stronger form of the curse of dimensionality for the class

$$
\tilde{F}_{d,r,p} = \{ f : \mathbb{R}^d \to \mathbb{R} \mid f \in C^r \text{ satisfies } (13)-(15) \},
$$

where

$$
\|f\| \leq 1, \quad \text{Lip}(f) \leq d^{-\frac{1}{2} + \frac{p}{r+1}} \sqrt{18\varepsilon \pi},
$$

$$
\forall k \leq r : \max_{\theta_1, \ldots, \theta_k \in S^{d-1}} \text{Lip}(D^{\theta_1} \ldots D^{\theta_k} f) \leq d^{-\frac{1}{2} + \frac{p(k+1)}{r+1}} r^k \left(\sqrt{18\varepsilon \pi}\right)^{k+1}.
$$

**Theorem 3.** For any $r \in \mathbb{N}$ and $p > 0$,

$$
n(\varepsilon, \tilde{F}_{d,r,p}) \geq (1 - \varepsilon) d^{pd/(r+1)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).
$$

Hence the curse of dimensionality holds for the class $\tilde{F}_{d,r,p}$. 

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Note that the classes $\tilde{F}_{d,r,p}$ are contained in the classes

$$C_d^r = \{ f \in C^r \mid \| D^\beta f \| \leq 1 \text{ for all } |\beta| \leq r \},$$

if $p < 1/2$ and $d$ is large enough. This holds if

$$d \geq \left( r^r (18 \pi)^{(r+1)/2} \right)^{1/(1/2-p)}.$$

From this we easily obtain the main result already stated in the introduction.

**Main Theorem.** For any $r \in \mathbb{N}$, there exists a constant $c_r \in (0, 1]$ such that

$$n(\varepsilon, C_d^r) \geq c_r (1 - \varepsilon) d^{d/(2r+3)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$

Hence the curse of dimensionality holds for the class $C_d^r$.

**Proof.** The case $d = 1$ is trivial since the initial error for the classes $C_d^r$ is again 1.

For $d \geq 2$, we know from Theorem 3 and the discussion thereafter that $n(\varepsilon, C_d^r) \geq (1 - \varepsilon) d^{pd/(r+1)}$ for all $p < 1/2$ if $d \geq d_0$, where $d_0 = d_0(r, p)$ is the right hand side of (16). This implies that for

$$\tilde{c}_{r,p} = d_0^{-pd_0/(r+1)},$$

which depends only on $r$ and $p$, we have

$$n(\varepsilon, C_d^r) \geq \tilde{c}_{r,p} (1 - \varepsilon) d^{pd/(r+1)} \quad \text{for all } d \geq 2.$$

The choice $p^* = (r + 1)/(2r + 3)$ yields the result with $c_r = \tilde{c}_{r,p^*}$. \qed

**Remark 1.** The reader might find it more natural to define classes of functions $F_{d,r}(D_d)$ that are defined only on $D_d \subset \mathbb{R}^d$. Not all such functions can be extended to smooth functions on $\mathbb{R}^d$, and even if they can be extended then the norm of the extended function could be much larger. Our lower bound results for functions defined on $\mathbb{R}^d$ can be also applied for functions defined on $D_d \subset \mathbb{R}^d$ and this makes them even stronger.

**Remark 2.** Note that the possibility of super-exponential lower bounds on the complexity depends on the definition of the Lipschitz constant. For the class

$$F_d = \left\{ f : [0, 1]^d \to \mathbb{R} \mid \sup_{x,y \in [0,1]^d} \frac{|f(x) - f(y)|}{\|x - y\|_\infty} \leq 1 \right\},$$

Sukharev [8] proved that for $n = m^d$ the midpoint rule is optimal with error $e_n = \frac{d}{2d + 2} n^{-1/d}$. Hence, roughly, $n(\varepsilon, F_d) \approx 2^{-d} \varepsilon^{-d}$ and the complexity is “only” exponential in $d$ for $\varepsilon < 1/2$. 13
Remark 3. We mention two results for the very small class

\[ F_d = C_d^\infty = \{ f \in C^\infty([0,1]^d) \mid \| D^\beta f \| \leq 1 \text{ for all } \beta \in \mathbb{N}_0^d \}. \]

O. Wojtaszczyk [10] proved that \( \lim_{d \to \infty} n(\varepsilon, F_d) = \infty \) for every \( \varepsilon < 1 \), hence the problem is not strongly polynomially tractable. It is still open whether the curse of dimensionality holds for this class \( F_d \). The same class \( F_d \) was studied for the approximation problem in [6]. For this problem the curse of dimensionality is present even if we allow algorithms that use arbitrary linear functionals.

References


