

# Appendix: Dini theorem for pseudo-Riemannian metrics

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## 1 Introduction

Consider a Riemannian or a pseudo-Riemannian metric  $g = (g_{ij})$  on a surface  $M^2$ . We say that a metric  $\bar{g}$  on the same surface is *projectively equivalent* to  $g$ , if every geodesic of  $\bar{g}$  is a reparametrized geodesic of  $g$ . In 1865 Beltrami [2] asked<sup>1</sup> to describe all pairs of projectively equivalent Riemannian metrics on surfaces. From the context it is clear that he considered this problem locally, in a neighbourhood of almost every point.

Theorem A below, which is the main result of this note, gives an answer to the following generalization of the question of Beltrami: we allow the metrics  $g$  and  $\bar{g}$  to be pseudo-Riemannian.

**Theorem A.** *Let  $g, \bar{g}$  be projectively equivalent metrics on  $M^2$ , and  $\bar{g} \neq \text{const} \cdot g$  for every  $\text{const} \in \mathbb{R}$ . Then, in the neighbourhood of almost every point there exist coordinates  $(x, y)$  such that the metrics are as in the following table.*

	Liouville case	Complex-Liouville case	Jordan-block case
$g$	$(X(x) - Y(y))(dx^2 \pm dy^2)$	$2\Im(h)dxdy$	$(1 + xY'(y))dxdy$
$\bar{g}$	$\left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right) \left(\frac{dx^2}{X(x)} \pm \frac{dy^2}{Y(y)}\right)$	$-\left(\frac{\Im(h)}{\Im(h)^2 + \Re(h)^2}\right)^2 dx^2$ $+ 2\frac{\Re(h)\Im(h)}{(\Im(h)^2 + \Re(h)^2)^2} dxdy$ $+ \left(\frac{\Im(h)}{\Im(h)^2 + \Re(h)^2}\right)^2 dy^2$	$\frac{1 + xY'(y)}{Y(y)^4} (-2Y(y)dxdy$ $+ (1 + xY'(y))^2 dy^2)$

where  $h := \Re(h) + i \cdot \Im(h)$  is a holomorphic function of the variable  $z := x + i \cdot y$ .

**Remark 1.** *It is natural to consider the metrics from the Complex-Liouville case as the complexification of the metrics from the Liouville case: indeed, in the complex coordinates  $z = x + i \cdot y$ ,  $\bar{z} = x - i \cdot y$ , the metrics have the form*

$$\begin{aligned} ds_g^2 &= -\frac{1}{4}(\bar{h}(z) - h(z))(dz^2 - d\bar{z}^2), \\ ds_{\bar{g}}^2 &= -\frac{1}{4}\left(\frac{1}{\bar{h}(z)} - \frac{1}{h(z)}\right)\left(\frac{dz^2}{h(z)} - \frac{d\bar{z}^2}{\bar{h}(z)}\right). \end{aligned}$$

**Remark 2.** *In the Jordan-block case, if  $dY \neq 0$  (which is always the case at almost every point, if the restriction of  $g$  to any neighborhood does not admit a Killing vector field), after a local coordinate change, the metrics  $g$  and  $\bar{g}$  have the form*

$$\begin{aligned} ds_g^2 &= (\tilde{Y}(y) + x)dxdy \\ ds_{\bar{g}}^2 &= -\frac{2(\tilde{Y}(y) + x)}{y^3}dxdy + \frac{(\tilde{Y}(y) + x)^2}{y^4}dy^2. \end{aligned}$$

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<sup>1</sup>Italian original from [2]: La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un'altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.

We see that the metrics from Complex-Liouville and Jordan-block cases always have signature  $(+, -)$ , and the metric  $g$  from the Liouville case has signature  $(+, +)$  or  $(-, -)$ , only if the sign “ $\pm$ ” is “ $+$ ”. In this case, the formulas from Theorem A are precisely the formulas obtained by Dini in [5].

We do not insist that we are the first to find these normal forms of projectively equivalent pseudo-Riemannian metrics. According to [1], a description of projectively equivalent metrics was obtained by P. Shirokov in [13]. Unfortunately, we were not able to find the reference [13] to check it. The result of Theorem A could be even more classical, see Remark 4.

Given two projectively equivalent metrics, it is easy to understand what case they belong to. Indeed, the  $(1, 1)$ -tensor  $G_j^i := \sum_{\alpha=1}^2 \bar{g}_{j\alpha} g^{i\alpha}$ , where  $g^{i\alpha}$  is inverse to  $g_{i\alpha}$ , has two different real eigenvalues in the Liouville case, two complex-conjugated eigenvalues in the Complex-Liouville case, and is (conjugate to) a Jordan-block in the Jordan-block case.

There exists an interesting and useful connection of projectively equivalent metrics with integrable systems.

Recall that a function  $F : T^*M^2 \rightarrow \mathbb{R}$  is called *an integral* of the geodesic flow of  $g$ , if  $\{H, F\} = 0$ , where  $H := \frac{1}{2} g^{ij} p_i p_j : T^*M^2 \rightarrow \mathbb{R}$  is the kinetic energy corresponding to the metric, and  $\{, \}$  is the standard Poisson bracket on  $T^*M^2$ . Geometrically, this condition means that the function is constant on the orbits of the Hamiltonian system with the Hamiltonian  $H$ . We say the integral  $F$  is *quadratic in momenta*, if for every local coordinate system  $(x, y)$  on  $M^2$  it has the form

$$F(x, y, p_x, p_y) = a(x, y)p_x^2 + b(x, y)p_x p_y + c(x, y)p_y^2 \quad (1)$$

in the canonical coordinates  $(x, y, p_x, p_y)$  on  $T^*M^2$ . Geometrically, the formula (1) means that the restriction of the integral to every cotangent space  $T_p^*M^2 \cong \mathbb{R}^2$  is a homogeneous quadratic function. Of course,  $H$  itself is an integral quadratic in the momenta for  $g$ . We will say that the integral  $F$  is *nontrivial*, if  $F \neq \text{const} \cdot H$  for all  $\text{const} \in \mathbb{R}$ .

**Theorem B.** *Suppose the metric  $g$  on  $M^2$  admits a nontrivial integral quadratic in momenta. Then, in a neighbourhood of almost every point there exist coordinates  $(x, y)$  such that the metric and the integral are as in the following table*

	Liouville case	Complex-Liouville case	Jordan-block case
$g$	$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))dxdy$
$F$	$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x) - Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$ ,	$p_x^2 - 2\frac{Y(y)}{1 + xY'(y)}p_x p_y$

where  $h := \Re(h) + i \cdot \Im(h)$  is a holomorphic function of the variable  $z := x + iy$ .

Indeed, as it was shown in [7, 8], and as it was essentially known to Darboux [4, §§600–608], if two metrics  $g$  and  $\bar{g}$  are projectively equivalent, then

$$I : TM^2 \rightarrow \mathbb{R}, \quad I(\xi) := \bar{g}(\xi, \xi) \left( \frac{\det(g)}{\det(\bar{g})} \right)^{2/3} \quad (2)$$

is an integral of the geodesic flow of  $g$ . Moreover, it was shown in [3], see Section 2.4 there, see also [10], the above statement is proven to be true<sup>2</sup> in the other direction: if the function (1) is an integral for the geodesic flow of  $g$ , then the metrics  $g$  and  $\bar{g}$  are projectively equivalent. Thus, Theorem A and Theorem B are equivalent. In this paper, we will actually prove Theorem B obtaining Theorem A as its consequence.

**Remark 3.** *The corresponding natural Hamiltonian problem on the hyperbolic plane has been recently treated in [11] following the approach used by Rosquist and Ugla [12].*

**Remark 4.** *The formulas that will appear in the proof are very close to that in §593 of [4]. Darboux worked over complex numbers and therefore did not care about whether the metrics are Riemannian or pseudo-Riemannian. For example, Liouville and Complex-Liouville case are the same for him. Moreover, in §594, Darboux gets the formulas that are very close to that of Jordan-block case, though he was interested in the Riemannian case only, and, hence, treated this “imaginary” case as not interesting.*

<sup>2</sup>with a good will, one also can attribute this result to Darboux

## 2 Proof of Theorem B (and, hence, of Theorem A)

If the metric  $g$  has signature  $(+, +)$  or  $(-, -)$ , Theorem A and, hence, Theorem B, were obtained by Dini in [5]. Below we assume that the metric  $g$  has signature  $(+, -)$ .

### 2.1 Admissible coordinate systems and Birkhoff-Kolokoltsov forms

Let  $g$  be a pseudo-Riemannian metric on  $M^2$  of signature  $(+, -)$ . Consider (and fix) two linear independent vector fields  $V_1, V_2$  on  $M^2$  such that

- $g(V_1, V_1) = g(V_2, V_2) = 0$  and
- $g(V_1, V_2) > 0$ .

Such vector fields always exist locally (and, since our result is local, this is sufficient for our proof).

We will say that a local coordinate system  $(x, y)$  is *admissible*, if the vector fields  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  are proportional to  $V_1, V_2$  with positive coefficient of proportionality:

$$V_1(x, y) = \lambda_1(x, y) \frac{\partial}{\partial x}, \quad V_2(x, y) = \lambda_2(x, y) \frac{\partial}{\partial y}, \quad \text{where } \lambda_i > 0, \quad i = 1, 2.$$

Obviously,

- admissible coordinates exist in a sufficiently small neighborhood of every point,
- the metric  $g$  in admissible coordinates has the form

$$ds^2 = f(x, y) dx dy, \quad \text{where } f > 0, \quad (3)$$

- two admissible coordinate systems in one neighbourhood are connected by

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} x_{new}(x_{old}) \\ y_{new}(y_{old}) \end{pmatrix}, \quad \text{where } \frac{dx_{new}}{dx_{old}} > 0, \quad \frac{dy_{new}}{dy_{old}} > 0. \quad (4)$$

**Lemma.** *Let  $(x, y)$  be an admissible coordinate system for  $g$ . Let  $F$  given by (1) be an integral for  $g$ . Then,  $B_1 := \frac{1}{\sqrt{|a(x, y)|}} dx$  ( $B_2 := \frac{1}{\sqrt{|c(x, y)|}} dy$ , respectively) is a 1-form, which is defined at points such that  $a \neq 0$  ( $c \neq 0$ , respectively). Moreover, the coefficient  $a$  ( $c$ , respectively) depends only on  $x$  ( $y$ , respectively), which in particular imply that the forms  $B_1, B_2$  are closed.*

**Remark 5.** *The forms  $B_1, B_2$  are not the direct analog of the ‘‘Birkhoff’’ 2-form introduced by Kolokoltsov in [6]. In a certain sense, they are the real analog of the different branches of the square root of the Birkhoff form.*

**Proof of the Lemma.** The first part of the statement, namely that the  $\frac{1}{\sqrt{|a|}} dx$  ( $\frac{1}{\sqrt{|c|}} dy$ , respectively) transforms as a 1-form under admissible coordinate changes is evident: indeed, after the coordinate change (4), the momenta transform as follows:  $p_{x_{old}} = p_{x_{new}} \frac{dx_{new}}{dx_{old}}$ ,  $p_{y_{old}} = p_{y_{new}} \frac{dy_{new}}{dy_{old}}$ . Then, the integral  $F$  in the new coordinates has the form

$$\underbrace{a \left( \frac{dx_{new}}{dx_{old}} \right)^2}_{a_{new}} p_{x_{new}}^2 + \underbrace{b \frac{dx_{new}}{dx_{old}} \frac{dy_{new}}{dy_{old}}}_{b_{new}} p_{x_{new}} p_{y_{new}} + \underbrace{c \left( \frac{dy_{new}}{dy_{old}} \right)^2}_{c_{new}} p_{y_{new}}^2.$$

Then, the formal expression  $\frac{1}{\sqrt{|a|}} dx_{old}$  ( $\frac{1}{\sqrt{|c|}} dy_{old}$ , respectively) transforms in

$$\frac{1}{\sqrt{|a|}} \frac{dx_{old}}{dx_{new}} dx_{new} \quad \left( \frac{1}{\sqrt{|c|}} \frac{dy_{old}}{dy_{new}} dy_{new}, \text{ respectively} \right),$$

which is precisely the transformation law of 1-forms.

Let us prove that the forms are closed. If  $g$  is given by (3), its Hamiltonian  $H$  is given by  $\frac{p_x p_y}{2f}$ , and the condition  $0 = \{H, F\}$  reads

$$\begin{aligned} 0 &= \left\{ \frac{p_x p_y}{2f}, a p_x^2 + b p_x p_y + c p_y^2 \right\} \\ &= \frac{1}{f} \left( p_x^3 (f a_y) + p_x^2 p_y (f a_x + f b_y + 2f_x a + f_y b) + p_y p_x^2 (f b_x + f c_y + f_x b + 2f_y) + p_y^3 (c_x f) \right), \end{aligned}$$

i.e., is equivalent to the following system of PDE:

$$\begin{cases} a_y = 0 \\ f a_x + f b_y + 2f_x a + f_y b = 0 \\ f b_x + f c_y + f_x b + 2f_y c = 0 \\ c_x = 0 \end{cases} \quad (5)$$

Thus,  $a = a(x)$ ,  $c = c(y)$ , which is equivalent to  $B_1 := \frac{1}{\sqrt{|a|}} dx$  and  $B_2 := \frac{1}{\sqrt{|c|}} dy$  are closed forms (assuming  $a \neq 0$  and  $c \neq 0$ ).  $\square$

**Remark 6.** For further use let us formulate one more consequence of the equations (5): if  $a \equiv c \equiv 0$  in a neighborhood of a point, then  $bf = \text{const}$  implying  $F \equiv \text{const} \cdot H$  in the neighborhood.

Assume  $a \neq 0$  ( $c \neq 0$ , respectively) at a point  $P_0$ . For every point  $P_1$  in a small neighbourhood  $U$  of  $P_0$  consider

$$x_{new} := \int_{\gamma: [0,1] \rightarrow U} B_1, \quad y_{new} := \int_{\gamma: [0,1] \rightarrow U} B_2, \quad \text{respectively} \quad (6)$$

$$\begin{aligned} &\gamma(0) = P_0 \\ &\gamma(1) = P_1 \end{aligned}$$

Locally, in the admissible coordinates, the functions  $x_{new}$  and  $y_{new}$  are given by

$$x_{new}(x) = \int_{x_0}^x \frac{1}{\sqrt{|a(t)|}} dt, \quad y_{new}(y) = \int_{y_0}^y \frac{1}{\sqrt{|c(t)|}} dt.$$

The new coordinates  $(x_{new}, y_{new})$  (or  $(x_{new}, y_{old})$  if  $c_{old} \equiv 0$ , or  $((x_{old}, y_{new})$  if  $a_{old} \equiv 0$ ) are admissible. In these coordinates, the forms  $B_1$  and  $B_2$  are given by  $\text{sign}(a_{old}) dx_{new}$ ,  $\text{sign}(c_{old}) dy_{new}$  (we assume  $\text{sign}(0) = 0$ ).

## 2.2 Proof of Theorem B

We assume that  $g$  of signature  $(+, -)$  on  $M^2$  admits a nontrivial quadratic integral  $F$  given by (1). Consider the matrix  $F^{ij} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ . It can be viewed as a  $(2,0)$ -tensor: if we change the coordinate system and rewrite the function  $F$  in the new coordinates, the matrix changes according to the tensor rule. Then,

$$\sum_{\alpha=1}^2 g_{j\alpha} F^{i\alpha}$$

is a  $(1,1)$ -tensor. In a neighborhood  $U$  of almost every point the Jordan normal form of this  $(1,1)$ -tensor is one of the following matrices:

$$\text{Case 1 } \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \text{Case 2 } \begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}, \quad \text{Case 3 } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where  $\lambda, \mu : U \rightarrow \mathbb{R}$ . Moreover, in view of Remark 6, there exists a neighborhood of almost every point such that  $\lambda \neq \mu$  in Case 1 and  $\mu \neq 0$  in Case 2. In the admissible coordinates, up to multiplication of  $F$  by  $-1$ , and renaming  $V_1 \leftrightarrow V_2$ , Case 1 is equivalent to the condition  $a > 0$ ,  $c > 0$ , Case 2 is equivalent to the condition  $a > 0$ ,  $c < 0$ , and Case 3 is equivalent to the condition  $c \equiv 0$ .

We now consider all three cases.

### 2.2.1 Case 1: $a > 0$ , $c > 0$ .

Consider the coordinates (6). In this coordinates,  $a = 1$ ,  $c = 1$ , and equations (5) are:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x + 2f_y = 0. \end{cases}$$

This system can be solved. Indeed, it is equivalent to

$$\begin{cases} (fb + 2f)_x + (fb + 2f)_y = 0, \\ (fb - 2f)_x - (fb - 2f)_y = 0, \end{cases}$$

which, after the change of coordinates  $x_{new} = x + y$ ,  $y_{new} = x - y$ , has the form

$$\begin{cases} (fb + 2f)_x = 0, \\ (fb - 2f)_y = 0, \end{cases}$$

implying  $fb + 2f = Y(y)$ ,  $fb - 2f = X(x)$ . Thus,  $f = \frac{Y(y) - X(x)}{4}$ ,  $b = 2\frac{X(x) + Y(y)}{Y(y) - X(x)}$ .

Finally, in the new coordinates, the metric and the integral have (up to a possible multiplication by a constant) the form

$$(X - Y)(dx^2 - dy^2) \\ 2\left(p_x^2 - \frac{X(x) + Y(y)}{X(x) - Y(y)}(p_x^2 - p_y^2) + p_y^2\right) = 4\frac{p_y^2 X(x) - p_x^2 Y(y)}{X(x) - Y(y)}.$$

Theorem B is proved under the assumptions of Case 1.

### 2.2.2 Case 2: $a > 0$ , $c < 0$ .

Consider the coordinates (6). In this coordinates,  $a = 1$ ,  $c = -1$ , and the equations (5) are:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x - 2f_y = 0. \end{cases}$$

We see that these conditions are the Cauchy-Riemann conditions for the complex-valued function  $fb + 2i \cdot f$ . Thus, for an appropriate holomorphic function  $h = h(x + i \cdot y)$ , we have  $fb = \Re(h)$ ,  $2f = \Im(h)$ . Finally, in a certain coordinate system the metric and the integral are (up to multiplication by constants):

$$2\Im(h)dxdy \quad \text{and} \quad p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y.$$

Theorem B is proved under the assumptions of Case 2.

### 2.2.3 Case 3: $a > 0$ , $c = 0$ .

Consider admissible coordinates  $x, y$ , such that  $x$  is the coordinate from (6). In these coordinates,  $a = 1$ ,  $c = 0$ , and the equations (5) are:

$$\begin{cases} (fb)_y + 2f_x = 0 \\ (fb)_x = 0 \end{cases}.$$

This system can be solved. Indeed, the second equation implies  $fb = -Y(y)$ . Substituting this in the first equation we obtain  $Y' = 2f_x$  implying

$$f = \frac{x}{2}Y'(y) + \widehat{Y}(y) \quad \text{and} \quad b = -\frac{Y(y)}{\frac{x}{2}Y'(y) + \widehat{Y}(y)}.$$

Finally, the metric and the integral are

$$\left(\widehat{Y}(y) + \frac{x}{2}Y'(y)\right)dxdy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{\widehat{Y}(y) + \frac{x}{2}Y'(y)}p_x p_y \quad (7)$$

Moreover, by the change  $y_{new} = \beta(y_{old})$  the metric and the integral (7) will be transformed to:

$$\left(\widehat{Y}(y)\beta' + \frac{x}{2}Y'(y)\right) dx dy \quad \text{and} \quad p_x^2 + \frac{Y(y)}{\widehat{Y}(y)\beta' + \frac{x}{2}Y'(y)} p_x p_y$$

Thus, by putting  $\beta(y) = \int_{y_0}^y \frac{1}{\widehat{Y}(t)} dt$ , we can make the metric and the integral to be

$$\left(1 + \frac{x}{2}Y'(y)\right) dx dy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{1 + \frac{x}{2}Y'(y)} p_x p_y.$$

Moreover, after the coordinate change  $x_{new} = \frac{x_{old}}{2}$  and dividing/multiplication of the metric/integral by 2, the metric and the integral have the form from Theorem B

$$(1 + xY'(y)) dx dy \quad \text{and} \quad p_x^2 - 2 \frac{Y(y)}{1 + xY'(y)} p_x p_y$$

Theorem B is proved.

## References

- [1] A. V. Aminova, *Projective transformations of pseudo-Riemannian manifolds. Geometry*, 9. J. Math. Sci. (N. Y.) **113** (2003), no. 3, 367–470.
- [2] E. Beltrami, *Resoluzione del problema: riportari i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette*, Ann. Mat., **1** (1865), no. 7, 185–204.
- [3] R. L. Bryant, G. Manno, V. S. Matveev, *A solution of a problem of Sophus Lie: Normal forms of 2-dim metrics admitting two projective vector fields*, accepted to Math. Ann, arXiv:0705.3592 .
- [4] G. Darboux, *Leçons sur la théorie générale des surfaces*, Vol. III, Chelsea Publishing, 1896.
- [5] U. Dini, *Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografiche di una superficie su un'altra*, Ann. Mat., ser.2, **3**(1869), 269–293.
- [6] V. N. Kolokoltsov, *Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial with respect to velocities*, Math. USSR-Izv. **21**(1983), no. 2, 291–306.
- [7] V. S. Matveev, P. J. Topalov, *Trajectory equivalence and corresponding integrals*, Regular and Chaotic Dynamics, **3** (1998), no. 2, 30–45.
- [8] V. S. Matveev, P. J. Topalov, *Geodesic equivalence of metrics on surfaces, and their integrability*, Dokl. Math. **60** (1999), no.1, 112-114.
- [9] V.S. Matveev, *Quantum integrability of the Beltrami-Laplace operator for geodesically equivalent metrics*. Dokl. Akad. Nauk **371**(2000), no. 3, 307–310.
- [10] V. S. Matveev, P. J. Topalov, *Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence*, Math. Z. **238**(2001), 833–866.
- [11] G. Pucacco, K. Rosquist, *(1+1)-dimensional separation of variables*, J. Math. Phys. (2007), **48**, 112903–112925.
- [12] K. Rosquist, C. Uggla, *Killing tensors in two-dimensional space-times with applications to cosmology*, J. Math. Phys. (1991), **32**, 3412–3422.
- [13] P. A. Shirokov, *Selected Works on Geometry*, Kazan Univ., Kazan (1966), 383–389.