SOME REMARKS ON NIJENHUIS BRACKET, FORMALITY, AND KÄHLER MANIFOLDS

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1. INTRODUCTION

A differentiable manifold M is said to be *formal* if the algebra $\wedge^*(M)$ of the differential forms on M is quasi isomorphic to its DeRham cohomology. (We recall that a morphism between Differential Graded Algebras is said to be a *quasi isomorphism* if it induces an isomorphism in cohomology and that two DGA's are said to be *quasi isomorphic* if they are equivalent with respect to the equivalence relation generated by quasi isomorphisms (cf. also [2]).)
It is well known that (cf. [6])

$$\boxed{(\wedge^*(M),\, d)\ \text{ is formal}} \Longrightarrow \boxed{\begin{array}{l}\text{from } H^*(M,\mathbb{R}) \text{ we can reconstruct}\\ \text{(via its minimal model, Postnikov towers etc...)}\\ \text{the whole rational}\\ \text{(i.e. all the cofinite) homotopy theory of } M\,.\end{array}}$$

One (actually, almost the only effective) way to get formality is to be able to produce a suitable derivation δ on $\wedge^*(M)$, $\delta : \wedge^k \to \wedge^{k+1}$ (for $k = 0, ..., n$), satisfying $\delta^2 = 0$ and such that *$d\delta$-lemma* holds, i.e. $(Ker\, d \cap Ker\, \delta) \cap (Im\, d + Im\, \delta) = Im\, d\delta$.

More precisely, the following general statements holds:

Theorem 1.1 (cf. [6])**.** *Let M be a smooth manifold with a derivation $\delta : \wedge^k \to \wedge^{k+1}$ (for $k = 0, ..., n$), satisfying $\delta^2 = 0$ such that $d\delta$-lemma holds. Then*

$$H(\wedge^*(M),\, d) = (Ker\, d \cap Ker\, \delta)/Im\, d\delta$$

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and so $(\wedge^*(M),\, d)$ and $(\wedge^*(M),\, \delta)$ are formal.

An example of such a situation is provided by Kähler manifolds: in this case, $\delta = d^c := J^{-1}dJ$, where J is the complex structure (cf. again [6]) .

We first show (Lemma 2.1, Remark 2.2) that the derivation δ satisfying properties above must be of the form $\delta = d_R := RdR^{-1}$, with $R \in End(TM)$ (i.e., R is a field of non degenerate linear transformations of the tangent spaces).

Then, we prove (Lemma 2.4) that the supercommutation of d and $\delta = d_R$ (which is a natural, essentially necessary condition to get a $d\delta$-lemma) amounts to $N_R \equiv 0$, N_R being the Nijenhuis tensor of R . Then, we are looking for sufficient conditions that ensure the dd_R-lemma holds. For R self adjoint with respect to a Riemannian metric, it is done in Section 3. For R compatible with an almost symplectic structure this is done in Section 4. Finally, we show that, if ${}^tR = -R$ and $det\, R \equiv 1$, then

$$N_R \equiv 0 \implies N_J \equiv 0$$

where J is the orthogonal component of R, in its polar decomposition and this also provides a new characterization of Kähler structures.

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2. Preliminary remarks: on the space of derivations

We begin with the following

Lemma 2.1. *Let M be a smooth compact manifold of dimension n and let $\delta \in End(\wedge^*(M))$ such that:*

a. $\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \delta\beta$ *and* $|\delta| = 1$ *, i.e.* $\delta : \wedge^p(M) \longrightarrow \wedge^{p+1}(M)$

b. $Ker\,\delta \cap \wedge^0(M) = \mathbb{R}$

c. $\delta^2 = 0$ *;*

then $R : X \mapsto \delta_X$ *belongs to* $End(TM)$ *and* $\delta = d_R := RdR^{-1}$ *(where, clearly,* $\delta_X : f \mapsto \delta f(X)$ *)*

Proof. The linearity of R is evident. By (b), R is nondegenerate. In order to prove $\delta = d_R$, let us note that for any $f \in \wedge^0(M)$, $X \in TM$, we have:

$$d_R f(X) = df(RX) = RXf = \delta_X f = (\delta f)(X)$$

i.e. δ coincides with d_R on $\wedge^0(M)$; this, together with (a), (c), is sufficient to insure $\delta \equiv d_R$. □

Remark 2.2. *Assume δ satisfies (a), (c) of lemma 2.1. Suppose*

- *the $d\delta$-lemma holds, i.e.:*

$$(Ker\,d \cap Ker\,\delta) \cap (Im\,d + Im\,\delta) = Im\,d\delta$$

- $d\delta + \delta d = 0$.

Then also (b) of lemma (2.1) *is fulfilled.*

Proof. Indeed, if $f \in Ker\,\delta \cap \wedge^0(M)$, $f \neq const$, then

$$0 \neq df \in (Ker\,d \cap Ker\,\delta) \cap (Im\,d + Im\,\delta)\,,$$

contradicting $df \notin Im\,d\delta$. □

For any $S \in End(TM)$, we define the *Nijenhuis tensor* of S as the element

$$N_S \in \wedge^2(M) \otimes TM$$

given by

$$N_S(X,\,Y) := [SX,\,SY] + S^2[X,\,Y] - S[SX,\,Y] - S[X,\,SY]\,;$$

It is known (and follows direct from definitions), that

- $N_{\mathbf{I}+S} = N_S$ (where $\mathbf{I} : TM \to TM$ is the identity)
- for any $\lambda \in C^\infty(M,\,\mathbb{R})$, $N_{\lambda\mathbf{I}} \equiv 0$
- if $R \in End(TM)$ then

$$N_{R^{-1}}(X,\,Y) = R^{-2}N_R(R^{-1}X,\,R^{-1}Y)\,.$$

Let V be a vector space. For any $L \in End(V)$ we consider

$$\tau(L) \in End(\wedge V^*)$$

defined as follows:

$$(\tau(L)(\alpha))(v_1,\,\dots,\,v_p) := \sum_{h=1}^{p} \alpha(v_1,\,\dots,L(v_h),\,\dots,\,v_p)\,. \tag{1}$$

We recall that a *Differential Graded Lie Algebra* (DGLA) is a graded vector space

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$$

together with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ and a degree one graded derivation d on $\mathfrak{g}$ in such a way that:

- $[\mathfrak{g}_j,\,\mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$

- for homogeneous elements a, b, c, we have:

$$\begin{aligned} [a,\, b] =&\quad -(-1)^{|a||b|}[b,\, a] \\ [a,\, [b,\, c]] =&\quad [[a,\, b],\, c] + (-1)^{|a||b|}[b,\, [a,\, c]] \qquad \text{Jacobi identity} \\ d[a,\, b] =&\quad [da,\, b] + (-1)^{|a|}[a,\, db]. \end{aligned}$$

- $d^2 = 0$

For example, there is a natural structure of $DGLA$ on $End(\wedge^*(M))$:
the gradation is obvious: $|P| = |P\alpha| - |\alpha|$,
and the bracket [,] and the derivation (we use the letter $\urcorner$ for it) are given by

- $[P,\, Q] := P \circ Q - (-1)^{|P||Q|} Q \circ P$,
- $\urcorner P := [d,\, P]$.

Let us recall the lemma (cf. e.g. [4]),

Lemma 2.3. *Let $R \in Aut(TM)$. Then,*

$$d_R = d + [\tau(S),\, d] - r(R),$$

where τ is defined by (1), $S := R - \mathbf{I}$, and $r(R)$ is a zero order differential operator quadratic in S defined as follows:

- *$r(R) \equiv 0$ on $\wedge^0(M)$*
- *for $\alpha \in \wedge^1(M)$ $r(R)(\alpha)(X,\, Y)) := \alpha(R^{-1}N_R(X,\, Y))$,*
- *extend to general α as skew-symmetric derivation.*

Proof. It is enough to prove the lemma for $f \in \wedge^0(M)$ and for $\alpha \in \wedge^1(M)$. If $f \in \wedge^0 M)$, we have:

$$d_R f(X) = df((\mathbf{I} + S)X) = df(X) + ([\tau(S),\, d])f(X)\,;$$

if $\alpha \in \wedge^1(M)$ we have first:

$$\begin{aligned} &([\tau(S),\, d]\alpha)(X,\, Y) = \\ &\qquad\qquad SX\alpha(Y) - SX\alpha(X) + \\ &\qquad\qquad \alpha(S[X,\, Y] - [SX,\, Y] - [X,\, SY])\,; \end{aligned}$$

then:

$$\begin{aligned} (d_R\alpha)(X,\, Y) =&(dR^{-1}\alpha)((\mathbf{I} + S)X,\, (\mathbf{I} + S)Y) = \\ &(I + S)X\alpha(Y) - (\mathbf{I} + S)Y\alpha(X) - \alpha((\mathbf{I} + S)^{-1}[(\mathbf{I} + S)X,\, (\mathbf{I} + S)Y]) \\ &X\alpha(Y) + SX\alpha(Y) - Y\alpha(X) - SY\alpha(X) - \alpha((\mathbf{I} + S)^{-1}[(\mathbf{I} + S)X,\, (\mathbf{I} + S)Y]) \\ &d\alpha(X,\, Y) + \alpha([X,\, Y]) + ([\tau(S),\, d]\alpha)(X,\, Y) + \\ &-\alpha(S[X,\, Y] - [SX,\, Y] - [X,\, SY]) - \alpha((\mathbf{I} + S)^{-1}[(\mathbf{I} + S)X,\, (\mathbf{I} + S)Y]) = \\ &d\alpha(X,\, Y) + ([\tau(S),\, d]\alpha)(X,\, Y) + \\ &\alpha((\mathbf{I} + S)^{-1}((\mathbf{I} + S)[X,\, Y] - (\mathbf{I} + S)S[X,\, Y] + (\mathbf{I} + S)[SX,\, Y] + (\mathbf{I} + S)[X,\, SY]) \\ &-\alpha((\mathbf{I} + S)^{-1}[(\mathbf{I} + S)X,\, (\mathbf{I} + S)Y]) = \\ &d\alpha(X,\, Y) + ([\tau(S),\, d]\alpha)(X,\, Y) - \alpha((\mathbf{I} + S)^{-1}N_S(X,\, Y) \end{aligned}$$

□

We will need the following

Lemma 2.4.

$$[d_R,\, d] = 0 \iff N_R = 0\,(\iff d_R = d + [\tau(S),\, d]).$$

Proof. Let us first show that d commutes with $d + [\tau(S),\, d]$. Since $d^2 = 0$, d commutes with itself. In order to show that d commutes with $[\tau(S),\, d] = 0$, we use the Jacobi identity:

$$[d,\, [\tau(S),\, d]] = [[d,\, \tau(S)],\, d] = -[d,\, [\tau(S),\, d]] \implies [d, [\tau(S),\, d]] = 0.$$

(In particular, the above observation shows the "$\Longleftarrow$" direction of the lemma)

In order to show that d commutes with $r(R)$ if and only if $N_R = 0$, we use that, for every $f \in \wedge^0(M)$, $X, Y \in TM$, we have,

$$[d,\, d_R]f(X,\, Y) = (r(R)df)(X,\, Y) = df(R^{-1}N_R(X,\, Y)).$$

Clearly, the right hand side vanishes for all X, Y if and only if $N_R \equiv 0$. ☐

Remark 2.5. *The previous lemma says that, in* $(End(\wedge^*(M)),\, [\, ,\,],\, \urcorner)$ *,*

$$\urcorner d_R = 0 \iff N_R = 0 \iff d - d_R = \urcorner\tau(S) \text{ i.e. } < d > = < d_R > \ .$$

Note also that:

$$[d,\, d_R] = 0$$
$$\Updownarrow$$
$$d_R \text{ satisfies the Maurer} - \text{Cartan equation } \urcorner d_R + \frac{1}{2}[d_R,\, d_R] = 0.$$

3. dd_R-LEMMA IN THE PRESENCE OF A RIEMANNIAN METRIC

Let g be a Riemannian metric on M. We denote by $*$ the Hodge-star operation. The next two lemmas says that when certain (natural) conditions on R are fulfilled, then the dd_R-lemma holds.

Lemma 3.1. *Let* $R \in Aut(TM)$ *such that:*

(1) $N_R = 0$ *(* $\overset{Lem.\ 2.4}{\iff} d_R = d + [\tau(S),\, d] \overset{Lem.\ 2.4}{\iff} [d,\, d_R] = 0$ *)*
(2) *there exists a Riemannian metric* g *on* M *such that*
    a. $[d_R,\, dd^*] = 0$
    b. $[d_R,\, d^*d] = 0$.

Then,

$$Ker\, d \cap Im\, d_R = Im\, dd_R$$

Proof. Set

$$\Delta_R := [d_R,\, d_R^*]\, .$$

Clearly,

$$[\Delta_R,\, d] = 0 = [\Delta,\, d_R]\, .$$

Note that $\Delta_R = R\tilde{\Delta}R^{-1}$, where $\tilde{\Delta}$ is the Laplacian operator with respect to $\tilde{g} = g(R\cdot,\, R\cdot)$. consider the Hodge decomposition with respect to Δ and Δ_R :

$$I = H + \Delta G$$
$$I = H_R + \Delta_R G_R\, .$$

Given $\alpha \in Ker\, d \cap Im\, d_R$ we have:

$$\alpha = H(\alpha) + dd^*G(\alpha)$$
$$\alpha = d_R d_R^* G_R(\alpha)\, .$$

Set $\gamma = d_R^* G_R(\alpha)$. Then,

$$\gamma = H(\gamma) + dd^*G(\gamma) + d^*dG(\gamma)$$

and so

$$d_R\gamma = d_R H(\gamma) + dd^*G(d_R\gamma)\, ,$$

i.e., $H(d_R\gamma) =$

$$d_R H(\gamma) = (d + [\tau(S),\, d])H(\gamma) = -d\tau(r)H(\gamma) = 0$$

and so:

$$\alpha = dd^*G(\alpha) = d_R d_R^* G_R(\alpha).$$

and finally
$\alpha = dd^*G(\alpha) = d_R d_R^* G_R(dd^*G(\alpha)) = dd_R G_R(d_R^* d^* G(\alpha))$, ☐

Corollary 3.2. *Let* $R \in End(TM)$ *such that:*

(1) $N_R = 0$ *(and so* $d_R = d + [\tau(S),\, d]$ *and* $[d,\, d_R] = 0$ *)*
(2) *there exists a Riemannian metric* g *on* M *such that*

a. $[d_R,\, dd^*] = 0$
b. $[d_R,\, d^*d] = 0$
c. $[d,\, d_R d_R^*] = 0$
d. $[d,\, d_R^* d_R] = 0\,;$

then the dd_R-lemma holds, i.e.

$$(Ker\, d \cap Ker\, d_R) \cap (Im\, d + Im\, d_R) = Im\, dd_R$$

4. dd_R−LEMMA IN THE ALMOST SYMPLECTIC SETTING

Let (M, κ) be an almost symplectic, $2n$-dimensional compact manifold. We consider

$$\mathcal{M}_\kappa(M) := \left\{ g \in \mathcal{R}iem(M) \,|\, d\mu(g) = \frac{\kappa^n}{n!} \right\}.$$

Recall (cf. [2]) that we can define the symplectic analog of the Hodge-star

$$\bigstar\, :\, \wedge^r(M) \longrightarrow \wedge^{2n-r}(M)$$

by means of the relation

$$\alpha \wedge \bigstar\beta = \kappa(\alpha,\, \beta)\frac{\kappa^n}{n!}$$

for $\alpha,\, \beta \in \wedge^r(M)$.

Analog to the Riemannian case, we consider on $\wedge^r(M)$:

$$d^\bigstar := (-1)^r \bigstar d \bigstar\,.$$

For any

$$g \in \mathcal{M}_\kappa(M)$$

there exists $R \in End(TM)$ such that

$$g(X,\, Y) = \kappa(R^{-1}X,\, Y)\,.$$

Clearly, R and R^{-1} are g-antisymmetric and $det\, R \equiv 1$;
on $\wedge^r(M)$ we have (cf. [3]):

$$\bigstar R^{-1} = (-1)^r * \;\; \text{i.e } *R = (-1)^r \bigstar \text{ and } R^{-1}* = \bigstar\,;$$

consequently:

$$d_R^* = -R^{-1} * d * R = -d^\bigstar.$$

We have the following

Lemma 4.1. *Assume*

- $N_R \equiv 0$
- $d_R[d,\, d^\bigstar] = 0 = [d,\, d^\bigstar]d_R$
- $d_{R^{-1}}[d,\, d^\bigstar] = 0 = [d,\, d^\bigstar]d_{R^{-1}}$

then, the dd_R-lemma holds.

Proof. We have:

$$[d,\, d_R d_R^*] = dd_R d_R^* - d_R d_R^* d = -d_R[d,\, d_R^*] = d_R[d,\, d^\bigstar]$$

and, similarly:

$$[d,\, d_R^* d_R] = -[d,\, d^\bigstar]d_R\,;$$

finally, we have:

$$[d_R,\, dd^*] = -d[d^*, d_R]$$

and

$$R^{-1}d[d^*,\, d_R]R = \pm d_{R^{-1}}[d^\bigstar,\, d].$$

Repeating this procedure with the other relation and applying Lemma 3.2 we obtain that dd_R−lemma holds. □

Remark 4.2. *If κ defines a symplectic structure, i.e. $d\kappa = 0$, then $[d,\, d^\bigstar] = 0$ (cf. e.g. [3]), and so we only need $N_R \equiv 0$:*

5. RELAXING THE CONDITION $J^2 = -\mathbf{I}$ IN THE DEFINITION OF KÄHLER MANIFOLD.

One of the equivalent definitions of the Kähler manifold is the following one: A Kähler manifold is a symplectic manifold (M, κ) equipped with $J \in End(TM)$ such that the bilinear form g defined by the equality $g(X, Y) := \kappa(X, JY)$ is a Riemannian metric and such that
I $J^2 = -\mathbf{I}$
II $N_J = 0$.

A lot of papers study the consequences of relaxing the second condition $N_J = 0$. In this case, the structure J is called an *almost complex structure*, and many papers are dedicated to almost complex structures satisfying additional conditions, see for example [8].

What about relaxing the first condition?

Theorem 5.1. *Let (M, κ) be an almost symplectic, $2n$-dimensional connected manifold; let again*

$$\mathcal{M}_\kappa(M) := \left\{ g \in \mathcal{R}iem(M) \,|\, d\mu(g) = \frac{\kappa^n}{n!} \right\} . \tag{2}$$

Assume there exists $g \in \mathcal{M}_\kappa(M)$ such that, representing g via κ by $R \in End(TM)$, i.e. for R satisfying

$$g(X, Y) = \kappa(RX, Y)\,,$$

we have

$$N_R \equiv 0\,.$$

Then, the orthogonal component J of R in its g-polar decomposition is $g-$scew-symmetric and satisfies

$$N_J \equiv 0\,.$$

Moreover, if $d\kappa = 0$, then (M, g, J) is a Kähler manifold.

Proof. The proof is organized as follows: we will first show that the orthogonal component J of R in its g-polar decomposition is actually a polynomial of R (we will also see that the polynomial is real and odd). The property $N_J = 0$ will then follow from $N_R = 0$ by [5]. The closedness of the form $g(J\cdot, \cdot)$ will require certain additional work.

We consider $-R^2 := -R \circ R$. It is clearly self adjoint and positively definite with respect to g; by (2) we have $\det(R^2) = \text{const}$. Then, it is semi-simple, and all its eigenvalues are positive by linear algebra.

We denote by $m(x)$ the number of different eigenvalues of $-R^2$ at $x \in M$ and by $\lambda_1(x)^2 > ... > \lambda_{m(x)}(x)^2$ ($\lambda_j > 0$, $1 \leq j \leq m(x)$) the eigenvalues of $-R^2$ at $x \in M$.

We say that a point $x \in M$ is *stable* if $m(x)$ is constant in a neighborhood of x. By [9, Lemma 4], the set of stable points is open and everywhere dense on M. Later, we will even show that all points are stable. We shall first work near a stable point x.

By [5, Lemma 6], the Nijenhuis tensor $N_{-R^2} = 0$. By [7], in the neighborhood of x there exists a coordinate system $\bar{x} = \left(\bar{x}_1 = (x_1^1, , ..., x_1^{2k_1}), ..., \bar{x}_m = (x_m^1, ..., x_m^{2k_m})\right)$ such that in this coordinate system the matrix of $-R^2$ is block diagonal, the dimensions of the blocks are $2k_1, ..., 2k_m$, and such that the *jth* block is λ_j^2 times the identity $2k_j \times 2k_j$-matrix:

$$-R^2 = \begin{pmatrix} \lambda_1^2 \cdot \mathbf{I}_{2k_1} & & \\ & \ddots & \\ & & \lambda_m^2 \cdot \mathbf{I}_{2k_m} \end{pmatrix} . \tag{3}$$

Moreover, the function λ_j does not depend on the variables x_i^ℓ for $j \neq i$.

This in particular implies that all eigenvalues λ_i are actually constant: indeed, from (3) we know that the determinant of $-R^2$ is the product $(\lambda_1)^{2k_1} \cdot ... \cdot (\lambda_m)^{2k_m}$. By assumption, the determinat is constant. Since the functions λ_i depend on its own variables, all functions λ_i must be constant. Then, all points must be stable as we claimed before.

Remark 5.2. *For further use let us note that, since the eigenspaces of R corresponding to different eigenvalues are orthogonal, in these coordinates the matrix of g is also block-diagonal with the same as in* (3) *dimensions of the blocks; by construction, the components of R are also orthogonal with the same dimensions of the blocks*

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_m \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_m \end{pmatrix}. \tag{4}$$

Let us now cook with the help of R the field of endomorphisms J such that it is the orthogonal component R in its $g-$polar decomposition. We take the polynomial $P(X) = a_{2m-1}X^{2m-1}+...+a_0$ of degree at most $2m-1$ such that its value at the points $X = \mathsf{i}\lambda_1, ..., \mathsf{i}\lambda_m$ is equal to i and such that its value at the points $X = -\mathsf{i}\lambda_1, ..., -\mathsf{i}\lambda_m$ is equal to $-\mathsf{i}$. From general theory it follows that such polynomial is unique (since the values in $2m$ points determine a unique polynomial of degree $2m-1$, see [1, §2 Ch. 1]). Since $P(\overline{X}) = \overline{P(X)}$ for $2m$ points $X = \pm\mathsf{i}\lambda_1, ..., \pm\mathsf{i}\lambda_m$, the coefficients of the polynomial are real. Since $P(-X) = -P(X)$ for $2m$ points $X = \pm\mathsf{i}\lambda_1, ..., \pm\mathsf{i}\lambda_m$, the polynomial is odd (i.e., all terms of even degree are zero).

We would like to point out that, since λ_i are constant, the coefficients of the polynomial are constant.

We now consider $J := P(R) = a_{2m-1}R^{2m-1} + ... + a_1 \cdot R$ (we understand R^r as $\underbrace{R \circ R \circ .... \circ R}_{r \text{ times}}$). Let us show that J is indeed the orthogonal component of R in its $g-$polar decomposition.

Evidently, the eigenvalues of J are $P(\pm\mathsf{i}\lambda_i) = \pm\mathsf{i}$, and the algebraic multiplicity of each eigenvalue coincides with its geometric multiplicity. Then, $J^2 = -\mathbf{I}$.

Now, since the polynomial P is even, the bilinear form $g(J\cdot, \cdot)$ is scew-symmetric. Indeed, all terms of the polynomial of even degree are zero, and for every term of odd degree we have

$$g(a_{2\ell-1}R^{2\ell-1}(U), V) = -g(a_{2\ell-1}R^{2\ell-2}(U), R(V)) = g(a_{2\ell-1}R^{2\ell-3}(U), R^2(V)) = ... = -g(U, a_{2\ell-1}R^{2\ell-1}(V))$$

(each time we transport one R to the right hand side we change the sign; all together we make odd number the sign change). Then, each term $g(a_{2\ell-1}R^{2\ell-1}\cdot, \cdot)$ is skew-symmetric implying $g(J\cdot, \cdot)$ is scew-symmetric as well.

Then, J is a $g-$orthogonal operator. Indeed,

$$g(JV, \underbrace{JU}_{U'}) = -g(J\underbrace{JU}_{U'}, V) = g(U, V) = g(V, U).$$

Now, the operator $R{\cdot}J = R{\cdot}P(R)$ is $g-$symmetric (impyling $R = SJ$ for a certain $g-$symmetric operator S). Indeed, arguing as above, we have

$$g(a_{2\ell-1}R^{2\ell}(U), V) = -g(a_{2\ell-1}R^{2\ell-1}(U), R(V)) = g(a_{2\ell-1}R^{2\ell-3}(U), R^2(V)) = ... = g(U, a_{2\ell-1}R^{2\ell}(V))$$

(this time we transport 2ℓ R's from left to right, so we change the sign even number of times). Finally, $J = P(R)$ satisfies the following properties:

- It is $g-$orthogonal,
- $R = SJ$ for a certain $g-$symmetric operator.

Thus, J is the orthogonal component of R in its $g-$polar decomposition.

Our goal is to show that (g, J) is a Kähler structure on M provided κ s closed. We already have seen that J is $g-$skew-symmetric. The property $N_J \equiv 0$ follows from [5, Lemma 6].

Let us now prove prove that the form $g(J\cdot, \cdot)$ is also closed. We will work locally, in a coordinate system $\bar{x}$ constructed above. Combining these with the form (3) of $-R^2$, we obtain that the matrix of J is given by

$$J = \begin{pmatrix} \frac{1}{\lambda_1}R_1 & & \\ & \ddots & \\ & & \frac{1}{\lambda_m}R_m \end{pmatrix} \tag{5}$$

Combining (3) and (4) we see that the matrix of $\kappa(\cdot, \cdot) := g(R\cdot, \cdot)$ (in the coordinate system $\bar{x}$ above) is given by the matrix

$$\kappa = \begin{pmatrix} \overset{1}{\kappa} & & \\ & \ddots & \\ & & \overset{m}{\kappa} \end{pmatrix} = \begin{pmatrix} -R_1 g_1 & & \\ & \ddots & \\ & & -R_m g_m \end{pmatrix}. \tag{6}$$

Then, by (5), the matrix of $g(J\cdot,\cdot)$ is

$$\begin{pmatrix} -Jg_1 & & \\ & \ddots & \\ & & -Jg_m \end{pmatrix} = \begin{pmatrix} -\frac{1}{\lambda_1} R_1 g_1 & & \\ & \ddots & \\ & & -\frac{1}{\lambda_m} R_m g_m \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_1}\overset{1}{\kappa} & & \\ & \ddots & \\ & & \frac{1}{\lambda_m}\overset{m}{\kappa} \end{pmatrix}. \tag{7}$$

In what follows we will us the convention

$$\bar{x} = \left(\bar{x}_1 = (x_1^1, , ..., x_1^{2k_1}), ..., \bar{x}_m = (x_m^1, ..., x_m^{2k_m})\right) = \left(y^1, ..., y^{2n}\right),$$

i.e., $y^1 := x_1^1, ..., y^{2k_1} := x_1^{2k_1}, y^{2k_1+1} := x_2^1, ..., y^{2n} := x_m^{2k_m}$.

Now we use that the differential of the form is given by

$$d\left(\sum_{p,q=1}^{2n} \kappa_{pq} dy^p \wedge dy^q\right) = \sum_{p,q,s=1}^{2n} \left(\tfrac{\partial}{\partial y^s}\kappa_{pq}\right) dy^s \wedge dy^p \wedge dy^q.$$

If the matrix of the form κ is as in (6), i.e., if

$$\kappa = \underbrace{\sum_{\alpha,\beta=1}^{2k_1} \overset{1}{\kappa}_{\alpha\beta} dx_1^\alpha \wedge dx_1^\beta}_{\overset{1}{\kappa}} + ... + \underbrace{\sum_{\alpha,\beta=1}^{2k_m} \overset{m}{\kappa}_{\alpha\beta} dx_m^\alpha \wedge dx_m^\beta}_{\overset{m}{\kappa}},$$

then, the differential of κ is

$$d\kappa = d\overset{1}{\kappa} + ... + d\overset{m}{\kappa} = \sum_{i=1}^{m}\left(\underbrace{\sum_{p=1}^{2n}\sum_{\alpha,\beta=1}^{2k_i}\left(\tfrac{\partial}{\partial y^p}\overset{i}{\kappa}_{\alpha\beta}\right) dy^p \wedge dx_i^\alpha \wedge dx_i^\beta}_{d\overset{i}{\kappa}}\right).$$

We see that the components of the differentials of $d\overset{i}{\kappa}$ and $d\overset{j}{\kappa}$ do not combine for $i \neq j$. Indeed, every component of $d\overset{i}{\kappa}$ is proportional to a certain $dy^p \wedge dx_i^\alpha \wedge dx_i^\beta$, and every component of $d\overset{j}{\kappa}$ is proportional to a certain $dy^p \wedge dx_j^\alpha \wedge dx_j^\beta$. Then, $d\kappa = 0$ implies $d\overset{i}{\kappa} = 0$ for all i.

Now, by (7), the form $g(J\cdot,\cdot)$ is given by

$$\tfrac{1}{\lambda_1}\overset{1}{\kappa} + ... + \tfrac{1}{\lambda_1}\overset{m}{\kappa}. \tag{8}$$

Since λ_i are constants as we explained above, and $d\overset{i}{\kappa} = 0$, then the differential of (8) vanishes. Thus, $g(J\cdot,\cdot)$ is closed as we claim. Theorem 5.1 is proved. □

Definition 5.3. *Let $(M\,\kappa)$ be a $2n$-dimensional (compact) symplectic manifold;*
*$R \in End(TM)$ is said to be κ-*calibrated *if*

$$g := \kappa(R\cdot,\,\cdot)$$

is a Riemannian metric such that $d\mu(g) = \frac{\kappa^n}{n!}$.

From Theorem 5.1 we immediately obtain

Corollary 5.4. *Let $(M\,,\kappa)$ be a $2n$-dimensional connected symplectic manifold.*
Then the following statements are equivalent:

- *$(M\,,\kappa)$ admits a Kähler structure g, J such that $\kappa(\cdot,\cdot) = g(J\cdot,\cdot)$.*

- *there exists* $R \in End(TM)$ *such that it is* κ*-calibrated and such that* $N_R \equiv 0$.

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