

# TWO-DIMENSIONAL SUPERINTEGRABLE METRICS WITH ONE LINEAR AND ONE CUBIC INTEGRAL.

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ABSTRACT. We describe all local Riemannian metrics on surfaces whose geodesic flows are superintegrable with one integral linear in momenta and one integral cubic in momenta.

We also show that some of these metrics can be extended to  $S^2$ . This gives us new examples of Hamiltonian systems on the sphere with integrals of degree three in momenta, and the first examples of superintegrable metrics of nonconstant curvature on a closed surface.

## 1. INTRODUCTION

**1.1. Definitions and statement of the problem.** Let  $M^2$  be a surface (i.e., 2-dimensional real manifold) equipped with a Riemannian metric  $g = (g_{ij})$ . The *geodesic flow* of the metric  $g$  is the Hamiltonian system on the cotangent bundle  $T^*M^2$  with the Hamiltonian  $H := \frac{1}{2}g^{ij}p_i p_j$ , where  $(x, y) = (x_1, x_2)$  is a local coordinate system on  $M^2$ , and  $(p_x, p_y) = (p_1, p_2)$  are the correspondent *momenta*, i.e., the dual coordinates on  $T^*M^2$ .

We say that a function  $F : T^*M^2 \rightarrow \mathbb{R}$  is an *integral* of the geodesic flow of  $g$ , if  $\{F, H\} = 0$ , where  $\{, \}$  is the canonical Poisson bracket on  $T^*M^2$ . We say that the integral is *polynomial in momenta of degree  $d$* , if in every local coordinate system  $(x, y, p_x, p_y)$  it has the form

$$F(x, y, p_x, p_y) = \sum_{i=0}^d a_i(x, y) p_x^{d-i} p_y^i, \quad (1.1)$$

For example, the Hamiltonian  $H$  itself is an integral quadratic in momenta. Integrals polynomial in momenta of degree 1 (3, resp.) will be called *linear* (resp. *cubic*) integrals.

The first main result of the present paper is a complete solution of the following problem:

**Problem A.** *Describe locally all two-dimensional Riemannian metrics admitting one integral  $L$  linear in momenta and one integral  $F$  cubic in momenta such that  $L, F$ , and  $H$  are functionally independent.*

Recall that functions  $L, F, H$  are *functionally independent* if there exists a point on  $T^*M$  such that at this point the differentials  $dL, dF, dH$  are linearly independent. For integrals polynomial in momenta, linear independence of the differentials of the integrals at one point implies linear independence of the differentials of the integrals at every point of a certain everywhere dense open subset (assuming the manifold is connected).

Recall that two-dimensional metrics whose geodesic flows admit three functionally independent integrals of a certain special form (in most cases the integrals are assumed to be polynomial in momenta of certain fixed degrees) are called *superintegrable*. Superintegrable metrics (and Hamiltonian systems in general) is nowadays a hot topic in mathematical

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physics and differential geometry, due to various applications and interesting mathematical structure lying behind. We suggest [K-K-M-W, K-K-M, Te-Wi, Da-Yp] for a discussion of superintegrable systems from the viewpoint of mathematical physics, and [Kr, Br-Ma-Ma] from the viewpoint of differential geometry.

If the metric is superintegrable with two linear integrals, it has constant curvature. The metrics that are superintegrable with two quadratic integrals (in addition to the energy integral), or one linear and one quadratic, were described (locally, in a neighbourhood of almost every point) in the classical work of Koenigs [Koe].

The next case should be “linear integral + cubic integral”, but the only result we found in this direction is due to Rañada [Ra], Gravel [Gr], and Marquette and Winternitz [Mar-Wi] and concerns the Hamiltonian systems such that the Hamiltonian  $H$  is the sum of the standard kinetic energy  $K = \frac{1}{2}(p_x^2 + p_y^2)$  and a potential energy  $V(x, y) \neq \text{const}$ . They assumed the existence of (functionally independent) linear and cubic integrals and proved that for such systems the cubic integral is actually the product of the linear integral and of an integral quadratic in momenta, i.e., such systems can be obtained via the Maupertuis’ transformation from the superintegrable systems constructed by Koenigs [Koe]. In particular, all known examples of metrics satisfying assumptions in Problem A above were in a certain sense trivial: the metric has a constant curvature, or the metric is superintegrable with one quadratic and one linear integral, and every cubic integral is a product of the integral linear in momenta and an integral quadratic in momenta.

## 1.2. Main result: local normal forms of metrics admitting one linear and one cubic integral.

**Theorem 1.1.** *Let  $g$  be a Riemannian metric on the 2-dimensional connected manifold  $M^2$ .*

*Suppose the geodesic flow of  $g$  admits a linear integral  $L$  and a cubic integral  $F$  such that  $L$ ,  $F$  and the Hamiltonian  $H$  are functionally independent. Then, locally near every point  $p$  such that  $L|_{T_p^*M^2} \not\equiv 0$  there exist coordinates  $(x, y)$  and a real function  $h(x)$  such that the metric  $g$  has the form  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  where  $h_x$  is the  $x$ -derivative, and the function satisfies one of the following **Principal** (ordinary differential) equations:*

$$\begin{aligned} (i) \quad & h_x \cdot (A_0 \cdot h_x^2 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) - (A_3 \cdot \frac{\sin(\mu \cdot x)}{\mu} + A_4 \cdot \cos(\mu \cdot x)) = 0 \\ (ii) \quad & h_x \cdot (A_0 \cdot h_x^2 - \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) - (A_3 \cdot \frac{\sinh(\mu \cdot x)}{\mu} + A_4 \cdot \cosh(\mu \cdot x)) = 0 \\ (iii) \quad & h_x \cdot (A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2) - (A_3 \cdot x + A_4) = 0 \end{aligned} \quad (1.2)$$

with  $\mu > 0$  in the first two cases.

*In all three cases the metric  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  is superintegrable with one linear integral  $L = p_y$  and one cubic integral. In the case (i) a cubic integral  $F = F(x, y; p_x, p_y)$  can be given by*

$$F = (C_+ \cdot e^{\mu y} + C_- \cdot e^{-\mu y}) \cdot (a_0(x) \cdot p_x^3 + a_2(x) \cdot p_x p_y^2) + (C_+ \cdot e^{\mu y} - C_- \cdot e^{-\mu y}) \cdot (a_1(x) \cdot p_x^2 p_y + a_3(x) \cdot p_y^3)$$

where  $C_+, C_-$  are arbitrary constants and  $a_i(x)$  are functions given by

$$\begin{aligned} a_0(x) &= A_0 \cdot h_x^3 \\ a_1(x) &= (-\mu \cdot A_0 \cdot h(x) + \frac{A_1}{2\mu}) \cdot h_x^2 \\ a_2(x) &= \frac{1}{2} \cdot (3A_0 \cdot h_x^2 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) \cdot h_x \\ a_3(x) &= \frac{1}{2\mu} \cdot (3A_0 \cdot h_x^2 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) \cdot h_{xx}. \end{aligned} \quad (1.3)$$

In the case (ii) a cubic integral can be given by

$$F = C_e \cdot \cos(\mu \cdot y + \phi_0) \cdot (a_0(x) \cdot p_x^3 + a_2(x) \cdot p_x p_y^2) + C_e \cdot \sin(\mu \cdot y + \phi_0) \cdot (a_1(x) \cdot p_x^2 p_y + a_3(x) \cdot p_y^3)$$

where  $C_e, \phi_0$  are constants (“amplitude and phase”) and  $a_i(x)$  are functions given by

$$\begin{aligned} a_0(x) &= A_0 \cdot h_x^3 \\ a_1(x) &= (\mu \cdot A_0 \cdot h(x) + \frac{A_1}{2\mu}) \cdot h_x^2 \\ a_2(x) &= \frac{1}{2} \cdot (3A_0 \cdot h_x^2 - \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) \cdot h_x \\ a_3(x) &= \frac{1}{2\mu} \cdot (3A_0 \cdot h_x^2 - \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) \cdot h_{xx}. \end{aligned} \tag{1.4}$$

In the case (iii) a cubic integral can be given by

$$\begin{aligned} F &= C_1 \cdot \left( a_0(x) \cdot p_x^3 + a_2(x) \cdot p_x p_y^2 + \frac{y}{2} \cdot (A_1 \cdot h_x^2 \cdot p_x^2 p_y + (A_1 \cdot h_x^2 + A_3) \cdot p_y^3) \right) \\ &+ C_2 \cdot \left( y \cdot a_0(x) \cdot p_x^3 + a_1(x) \cdot p_x^2 p_y + y \cdot a_2(x) \cdot p_x p_y^2 + a_3(x) \cdot p_y^3 \right. \\ &\quad \left. + \frac{y^2}{4} \cdot (A_1 \cdot h_x^2 \cdot p_x^2 p_y + (A_1 \cdot h_x^2 + A_3) \cdot p_y^3) \right) \end{aligned} \tag{1.5}$$

where  $C_1, C_2$  are constants and  $a_i(x)$  are functions given by

$$\begin{aligned} a_0(x) &= A_0 \cdot h_x^3 \\ a_1(x) &= -A_0 \cdot h_x^2 \cdot h(x) \\ a_2(x) &= \frac{1}{2} \cdot (3 \cdot h_x^2 \cdot A_0 - A_1 \cdot h(x) + A_2) \cdot h_x \\ a_3(x) &= -\frac{1}{4} \cdot (4A_0 \cdot h_x^2 \cdot h(x) + A_3 \cdot x^2 + 2A_4 \cdot x). \end{aligned} \tag{1.6}$$

Moreover, in the case when the metric  $g$  has non-constant curvature every cubic integral is a linear combination  $F + C_{L3} \cdot L^3 + C_{LH} \cdot L \cdot H$  where  $F$  is given by the above formula (according to the case (i)–(iii)) and  $C_{L3}, C_{LH}$  are constants. In particular, in the non-constant curvature case the space of cubic integrals of our metrics has dimension 4.

**Remark 1.1. Uniqueness of the equation.** We show in Theorem 4.1 that in the case when the curvature of our metric  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  is non-constant the equation (1.2) on the function  $h(x)$  is *unique* up to a constant factor. On the other hand, in Theorem 3.1 we describe possible equations of the form (1.2) for whose the metric  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  has constant curvature.

Thus, Theorems 1.1, 3.1, and 4.1 give a complete answer to the Problem A above.

**Remark 1.2. Known special case: Darboux-superintegrable metrics.** We call a metric  $g$  on  $M^2$  *Darboux-superintegrable*, if it has non-constant curvature and the geodesic flow of the metric admits at least four linear independent integrals quadratic in momenta. For example, such is the metric  $(x^2 + y^2 + 1) \cdot (dx^2 + dy^2)$  on  $\mathbb{R}^2$  (see for example [Ma1, §4]).

Darboux-superintegrable metrics are well understood. Locally, they were described already by Koenigs [Koe]. In particular, Koenigs has shown that every Darboux-superintegrable metric admits a linear integral. Then, it also admits cubic integrals, namely the products of the linear integral and the quadratic integrals. Therefore, our Theorem 1.1 applies. In particular, in the appropriate local coordinates  $(x, y)$  the metric has the form  $\frac{1}{h_x^2}(dx^2 + dy^2)$  such that  $h(x)$  is a solution of the Principle equation (1.2), (i)–(iii).

The formulas above show that if the coefficient  $A_0$  vanishes, then a generic cubic integral  $F + C_{L3} \cdot L^3 + C_{LH} \cdot L \cdot H$  is the product of the linear integral and a function quadratic in

momenta which must automatically be an integral. Then, the metrics corresponding to  $A_0 = 0$  are Darboux-superintegrable or are of constant curvature. The uniqueness of the Principle equation (see the previous remark) shows that the converse assertion is also true. Thus we obtain the following characterisation: *Under the hypotheses of Theorem 1.1 the metric  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  of non-constant curvature is Darboux-superintegrable if and only if the parameter  $A_0$  vanishes.*

**Remark 1.3.** With the help of a computer algebra software, for example with Maple<sup>®</sup>, it is easy to check that the functions  $F$  from Theorem 1.1 are indeed integrals: the condition  $\{H, F\} = 0$  is equivalent to 5 ODEs of at most 3rd order on the function  $h$ ; these ODEs are identically fulfilled for the function  $h$  satisfying the corresponding equation (1.2), since they are algebraic corollaries of the corresponding equation and its first two derivatives. We will of course explain how we constructed the integrals (in Section 2) since we also need to show that we constructed all such metrics. Moreover, the idea of the construction will be used in the proof of other statements of Theorem 1.1, in particular in the proof that the dimension of the space of the integrals is 4. Moreover, we believe that the idea of our construction could also be used for constructing higher order superintegrable cases, see Problem 1 in the conclusion.

**1.3. Second main result: Examples of metrics on the 2-sphere admitting linear and cubic integrals.** The problem of finding and describing *global* integrable Hamiltonian systems, i.e., those whose configuration space is a compact manifold, is one of the central topics in the classical mechanics. The version of this problem in our context is as follows:

**Problem B.** *Understand what Riemannian metrics on the 2-sphere  $S^2$  admit one integral  $L$  linear in momenta and one integral  $F$  cubic in momenta such that  $L, F$ , and the Hamiltonian  $H$  are functionally independent.*

Note that other oriented closed surfaces can not admit superintegrable metrics. Indeed, if the metric is superintegrable, all geodesics are closed which is possible on the sphere and on  $\mathbb{R}P^2$  only.

It is known that the metric of constant curvature of the sphere do admit (linearly independent) linear and cubic integrals. So the nontrivial part of the Problem B is whether there are other metrics on the 2-sphere admitting an integral  $L$  linear in momenta and an integral  $F$  cubic in momenta such that  $L, F$ , and  $H$  are functionally independent.

For the integrals of lower degrees, the answer is negative. Indeed, the existence of two functionally independent linear integrals implies, even locally, that the metric has constant curvature. By Kiyohara [Ki], the existence of three functionally independent quadratic integrals (energy integral + two additional integrals) on the 2-sphere implies that the metric is of constant curvature. From this result, it also follows that the existence of (functionally independent) linear and two quadratic integrals implies that the metric of the 2-sphere has constant curvature. Because of these results (and absence of examples of superintegrable systems with higher degree integrals), it was generally believed that no polynomially superintegrable metric exists on a closed surface of nonconstant curvature.

In the present paper, we construct the first examples of smooth (even analytic) metrics of nonconstant curvature on the 2-sphere whose geodesic flows admit integrals  $L$  linear in momenta and integrals  $F$  cubic in momenta such that  $L, F$ , and  $H$  are functionally independent. The construction is in Section 6. We show that for certain values of parameters the

metrics we constructed in Theorem 1.1 and the integrals of these metrics can be smoothly extended to the sphere. More precisely (in the notation of Theorem 1.1), if the function  $h(x)$  fulfills the equation (1.2) (ii) and the condition  $h'(x_0) > 0$  at some point  $x_0$  whereas the real parameters  $\mu > 0, A_0, \dots, A_4$  satisfy inequalities  $A_0 > 0, \mu \cdot A_4 > |A_3|$  then the metric  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  smoothly extends to the sphere  $S^2$  together with the linear integral  $L = p_y$  and the cubic integral  $F$  given by (1.4).

We conject that *these are all examples of metrics on the sphere superintegrable with one linear and one cubic integral.*

Our examples are also interesting from other points of view. Indeed, every metric from these examples admit an integral cubic in momenta that not the product of a linear and a quadratic integral. The problem of constructing such metrics is very classical and, in a certain extend, was stated by Jacobi, Darboux, Cauchy, Whittaker, see also [Bo-Ko-Fo, Bo-Fo]. There are only very few examples of such metrics on closed surfaces: constant curvature metrics, metrics constructed via Maupertuis' transformation from the Goryachev-Chaplygin case of rigid body motion and their generalizations due to Goryachev [Go], metrics constructed by Selivanova [Se], by Kiyohara [Ki2], and by Dullin et al [Du-Ma] (see also [Du-Ma-To] and [Ma-Sh]). Note that the analogous question for the quadratic integrals is completely solved, see [Kol, Ki, Bo-Ma-Fo, Ma].

Moreover, all geodesics of the metrics we constructed are closed (since it is always the case for superintegrable metrics), so the examples are also examples of the so-called Zoll surfaces.

**1.4. Additional result: Special case of Kruglikov's "big gap" conjecture.** In [Kr, §12], Kruglikov has shown that the dimension of the space of cubic integrals (of the geodesic flow of a 2D-metric) is at most 10; the dimension 10 is achieved only by the metrics of constant curvature. He also has shown that the second largest dimension is at most 7 (see [Kr, Theorem 8]), and conjectured that the gap between the largest and the second largest possibilities for the dimension of the space of cubic integrals is even bigger: he writes that *it seems that the next realized dimension after 10 is 4.*

We will prove this conjecture (see Theorem 5.1) under the additional assumption that the metric admits a Killing vector field. Note that this assumption does not look too artificial, since it is expected that metrics with many polynomial integrals admit Killing vector fields. For example, by the classical result of Koenigs mentioned above, metrics admitting four (= the second largest dimension) linearly independent integrals that are quadratic in momenta admit Killing vector fields.

More precisely, we will prove that, if a 2D metric is superintegrable with one linear and one cubic integral ( $L$  and  $F$ ) and has non-constant curvature, then, locally, the space of cubic integrals is precisely 4-dimensional. In particular, in addition to the integrals  $L^3, F, L \cdot H$  we always construct one more cubic integral  $F_2$  that is linearly independent of  $L^3, F$ , and  $L \cdot H$ .

## 2. PRINCIPLE EQUATION AND OVERVIEW OF THE PROOF OF THEOREM 1.1

**2.1. How we found the metrics: scheme of the proof of Theorem 1.1.** It is well-known (see for example [Da, §592], or [Bo-Ma-Fo]) that every pair  $(g, L)$ , where  $g$  is a Riemannian metric, and  $L$  is an integral linear in momenta, is given in appropriate coordinates in a neighbourhood of every point such that  $L \neq 0$  by the formulas

$$g = \lambda(x)(dx^2 + dy^2) \quad \text{and} \quad L = p_y. \quad (2.1)$$

The natural “naive” method to solve the Problem A would be to write the condition  $\{H, F\} = 0$ , where  $H = \frac{p_x^2 + p_y^2}{2\lambda(x)}$  and  $F := a_0(x, y)p_x^3 + a_1(x, y)p_x^2p_y + a_2(x, y)p_xp_y^2 + a_3(x, y)p_y^3$ , as the systems of PDE on the unknown function  $\lambda$  of one variable and unknown functions  $a_i$  of two variables, and to try to solve it. Unfortunately, by this method we obtain a system of 5 nonlinear PDE on 5 unknown functions  $\lambda, a_0, a_1, a_2, a_3$ , which is completely intractable<sup>1</sup>.

In order to solve the problem, we used a trick that allowed us to reduce the problem to solving systems of ODE (instead of PDE). A similar trick was recently used in [Ma2].

The main observation is the following: the Poisson bracket of the linear integral  $L$  and of a cubic integral  $F$  is

- an integral (because of the Jacobi identity), and
- is cubic in momenta (because each term in the sum  $\{L, F\} = \partial_x F \partial_{p_x} L + \partial_y F \partial_{p_y} L - \partial_x L \partial_{p_x} F - \partial_y L \partial_{p_y} F$  is cubic in momenta).

Thus, the mapping  $\mathcal{L} : F \mapsto \{L, F\}$  is a linear homomorphism. By [Kr], the space of cubic integrals is finite- (at most, 10-)dimensional. Let us now consider the eigenvalues of the mapping  $\mathcal{L}$ . Clearly, 0 is an eigenvalue of  $\mathcal{L}$ , whose eigenvectors are  $A_3 \cdot L^3 + A_1 \cdot L \cdot H$ , where  $A_1, A_3 \in \mathbb{R}$ . The following two cases are possible:

**Case 1: The mapping  $\mathcal{L}$  has an eigenvalue  $\mu \neq 0$ .** Then, there exists a cubic integral  $F$  such that  $\{L, F\} = \mu \cdot F$ . We allow  $\mu$  to be a complex number, and  $F$  to be complex valued function, i.e.,  $F = F_1 + iF_2$  for real-valued cubic integrals  $F_1$  and  $F_2$ .

In the coordinates such that  $(g, L)$  are given by (2.1), we have

$$\{L, F\} = \partial_y F = \partial_y a_0(x, y) \cdot p_x^3 + \partial_y a_1(x, y) \cdot p_x^2 p_y + \partial_y a_2(x, y) \cdot p_x p_y^2 + \partial_y a_3(x, y) \cdot p_y^3,$$

so that the condition  $\{L, F\} = \mu \cdot F$  is equivalent to the system  $\partial_y a_i(x, y) = \mu \cdot a_i(x, y)$ ,  $i = 0, \dots, 3$ . Then,  $a_i(x, y) = \exp(\mu y) \cdot a_i(x)$  for certain (complex valued in the general case) functions  $a_i(x)$  of one variable  $x$ . Then, all unknown functions in the equation  $\{H, F\} = 0$  are functions of the variable  $x$  only, i.e., the condition  $\{H, F\} = 0$  is a system of ODE (depending on the parameter  $\mu$ ). Finally, the condition  $\{H, F\} = 0$  is equivalent to 5 ODE on 5 unknown functions of one variable  $x$ : four unknown functions  $a_i(x)$  and  $\lambda(x)$ . Working with this system of ODE, we partially integrate it and reduce it to one ODE of the first order (essentially, the first equation of (1.2) for  $\mu \in \mathbb{R}$  and the second equation of (1.2) for  $\mu \in i \cdot \mathbb{R}$ ). Further in §4.1 we shall show that the assumption that the function  $h_x$  is real implies that  $\mu$  is real or pure imaginary).

**Remark 2.1.** In this way, we obtain  $\lambda(x)$  which is a priori a complex-valued function; for our problem, only real-valued  $\lambda$ 's are of interest. We shall see in §4.1 that  $\lambda$  is real if and only if  $\mu$  is real or purely imaginary.

**Case 2: The mapping  $\mathcal{L}$  has only one eigenvalue, namely zero.** Since in our setting the space of cubic integrals is at least three-dimensional, there exists an integral  $F$  linear independent of  $L^3$  and  $L \cdot H$  such that  $\{L, F\} = \frac{A_3}{2} \cdot L^3 + A_1 \cdot L \cdot H$  for certain constants  $A_1, A_3$ .

<sup>1</sup>This “naive” approach to this problem was tried without success by many experts in superintegrable systems (private communications by Marquette, Rañada, Winternitz).

In the coordinates such that  $(g, L)$  is given by (2.1), the condition  $\{L, F\} = \frac{A_3}{2} \cdot L^3 + A_1 \cdot L \cdot H$  reads

$$\begin{aligned} & \{p_y, a_0(x, y)p_x^3 + a_1(x, y)p_x^2p_y + a_2(x, y)p_xp_y^2 + a_3(x, y)p_y^3\} \\ &= \partial_y a_0(x, y) \cdot p_x^3 + \partial_y a_1(x, y) \cdot p_x^2p_y + \partial_y a_2(x, y) \cdot p_xp_y^2 + \partial_y a_3(x, y) \cdot p_y^3 \\ &= \frac{A_3}{2} \cdot p_y^3 + \frac{A_1}{2\lambda(x)} \cdot (p_x^2 + p_y^2) \cdot p_y, \end{aligned}$$

and is equivalent to the system  $\partial_y a_0(x, y) = 0$ ,  $\partial_y a_1(x, y) = \frac{A_1}{2\lambda(x)}$ ,  $\partial_y a_2(x, y) = 0$ ,  $\partial_y a_3(x, y) = \frac{A_3}{2} + \frac{A_1}{2\lambda(x)}$ . Then,

$$F = a_0(x) \cdot p_x^3 + a_1(x) \cdot p_x^2p_y + a_2(x) \cdot p_xp_y^2 + a_3(x) \cdot p_y^3 + \frac{y}{2} \cdot \left( A_3 \cdot p_y^3 + A_1 \cdot p_y \cdot \frac{p_x^2 + p_y^2}{\lambda(x)} \right). \quad (2.2)$$

We again see that all unknown functions in the equation  $\{H, F\} = 0$  are functions of the variable  $x$  only, i.e., the condition  $\{H, F\} = 0$  is a system of 5 ODE (depending on the parameters  $A_1, A_3, y_0$ ) on 5 unknown functions of one variable  $x$ :  $a_i(x)$  and  $\lambda(x)$ . Working with this system of ODE, we partially integrate it and reduce it to one ODE of the first order (Equation (iii) of (1.2)), which is in a certain sense a degenerate case of the corresponding ODE we obtained in Case 1.

**2.2. Case 1 ( $\mu \neq 0$ ) in the proof of Theorem 1.1.** For convenience in further computation, we write the metric  $g$  in the form

$$g = \frac{dx^2 + dy^2}{h^2} \quad (2.3)$$

for some function  $h = h(x)$ , where  $h_x = \frac{dh(x)}{dx}$ . Then  $H = \frac{h_x^2}{2} \cdot (p_x^2 + p_y^2)$  and the linear integral is  $L := p_y$ .

We assume (see §2.1) that there exists a complex-valued cubic integral of the form

$$F = \exp(\mu \cdot y) \cdot a_0(x) \cdot p_x^3 + \exp(\mu \cdot y) \cdot a_1(x) \cdot p_x^2p_y + \exp(\mu \cdot y) \cdot a_2(x) \cdot p_xp_y^2 + \exp(\mu \cdot y) \cdot a_3(x) \cdot p_y^3,$$

where  $a_i$  are smooth complex-valued functions of one real variable  $x$ .

Then, the condition  $\{F, H\} = 0$  reads

$$\begin{aligned} \{F, H\} &= h_x \cdot \exp(\mu \cdot y) \cdot (h_x \cdot a_0(x)_x - 3 \cdot a_0(x) \cdot h_{xx}) \cdot p_x^4 \\ &+ h_x \cdot \exp(\mu \cdot y) \cdot (-2 \cdot a_1(x) \cdot h_{xx} + h_x \cdot \mu \cdot a_0(x) + h_x \cdot a_1(x)_x) \cdot p_x^3 \cdot p_y \\ &+ h_x \cdot \exp(\mu \cdot y) \cdot (h_x \cdot \mu \cdot a_1(x) - 3 \cdot a_0(x) \cdot h_{xx} - a_2(x) \cdot h_{xx} + h_x \cdot a_2(x)_x) \cdot p_x^2 \cdot p_y^2 \\ &+ h_x \cdot \exp(\mu \cdot y) \cdot (h_x \cdot \mu \cdot a_2(x) + h_x \cdot a_3(x)_x - 2 \cdot a_1(x) \cdot h_{xx}) \cdot p_x \cdot p_y^3 \\ &+ \frac{1}{2} h_x \cdot \exp(\mu \cdot y) \cdot (-a_2(x) \cdot h_{xx} + h_x \cdot \mu \cdot a_3(x)) \cdot p_y^4, \end{aligned} \quad (2.4)$$

where subscripts  $a_0(x)_x, h_x$  mean derivation in  $x$ , and  $h_{xx}$  is the second derivative. Since the monomials  $p_x^{4-i} p_y^i$  form a basis of homogeneous polynomials of degree 4, every line in (2.4) should vanish. This gives us a system of 5 ODEs on 5 functions  $h(x), a_0(x), \dots, a_3(x)$ : each line of (2.4) corresponds to one ODE. Subsequently solving the first three of them and resolving  $a_3(x)$  from the last one we obtain

$$\begin{aligned} a_0(x) &= A_0 \cdot h_x^3 \\ a_1(x) &= (-\mu \cdot A_0 \cdot h(x) + \frac{A_1}{2\mu}) \cdot h_x^2 \\ a_2(x) &= \frac{1}{2} \cdot (-A_1 \cdot h(x) + \mu^2 \cdot A_0 \cdot h(x)^2 + 3 \cdot h_x^2 \cdot A_0 + A_2) \cdot h_x \\ a_3(x) &= \frac{1}{2\mu} \cdot (3 \cdot h_x^2 \cdot A_0 - A_1 \cdot h(x) + \mu^2 \cdot A_0 \cdot h(x)^2 + A_2) \cdot h_{xx}, \end{aligned} \quad (2.5)$$

with some constants  $A_0, A_1, A_2$ . Substituting in the remaining equation  $(\dots)p_x \cdot p_y^3$ , we obtain the following non-linear ODE of order 3 on  $h(x)$ :

$$\begin{aligned} & (3 \cdot A_0 \cdot h_x^2 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) \cdot h_{xxx} + \\ & + 6 \cdot h_{xx}^2 \cdot h_x \cdot A_0 + (6 \cdot \mu^2 \cdot A_0 \cdot h(x) - 3 \cdot A_1) \cdot h_x \cdot h_{xx} + \\ & + 3 \cdot \mu^2 \cdot A_0 \cdot h_x^3 + (\mu^4 \cdot A_0 \cdot h(x)^2 - \mu^2 \cdot A_1 \cdot h(x) + \mu^2 \cdot A_2) \cdot h_x = 0. \end{aligned} \quad (2.6)$$

By direct calculations we see that the equation (2.6) can be written in the form

$$\left( \frac{d^2}{dx^2} + \mu^2 \right) (h_x \cdot (h_x^2 \cdot A_0 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2)) = 0. \quad (2.7)$$

Therefore this equation is equivalent to the equation

$$h_x \cdot (h_x^2 \cdot A_0 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) - \left( A_3 \cdot \frac{\sin(\mu x)}{\mu} + A_4 \cdot \cos(\mu x) \right) = 0 \quad (2.8)$$

in the sense that  $h(x)$  satisfies the equation (2.6) if and only if it satisfies (2.8) with the same constant  $A_0, A_1, A_2 \in \mathbb{C}$  and some constants  $A_3, A_4 \in \mathbb{C}$ . Later, in §2.5 (see Theorem 4.2) we shall show that only real  $A_0, A_1, A_2, A_3, A_4$  are interesting for our purposes.

**2.3. Case 2:**  $\mu = 0$ . We proceed as we explained in §2.1: we write the metric in the form  $g = \frac{1}{h_x^2} \cdot (dx^2 + dy^2)$ , so that now  $H = \frac{h_x^2}{2} \cdot (p_x^2 + p_y^2)$ , and then substitute (2.2) in the condition  $\{F, H\} = 0$ . We obtain

$$\begin{aligned} \{F, H\} &= h_x \cdot (h_x \cdot a_0(x)_x - 3 \cdot a_0(x) \cdot h_{xx}) \cdot p_x^4 \\ &+ h_x \cdot (h_x \cdot a_1(x)_x - 2 \cdot a_1(x) \cdot h_{xx}) \cdot p_x^3 \cdot p_y \\ &+ h_x \cdot \left( \frac{1}{2} \cdot A_1 \cdot h_x^3 - 3 \cdot a_0(x) \cdot h_{xx} - a_2(x) \cdot h_{xx} + h_x \cdot a_2(x)_x \right) \cdot p_x^2 \cdot p_y^2 \\ &+ h_x \cdot (h_x \cdot a_3(x)_x - 2 \cdot a_1(x) \cdot h_{xx}) \cdot p_x \cdot p_y^3 \\ &+ h_x \cdot (A_1 \cdot h_x^3 + A_3 \cdot h_x - 2 \cdot a_2(x) \cdot h_{xx}) \cdot p_y^4, \end{aligned} \quad (2.9)$$

with the same constants  $A_1, A_3$  as in (2.2). This time we can subsequently resolve all functions  $a_0(x), \dots, a_3(x)$  from the equations and obtain

$$\begin{aligned} a_0(x) &= A_0 \cdot h_x^3 \\ a_1(x) &= \frac{1}{2} \cdot \tilde{A}_1 \cdot h_x^2 \\ a_2(x) &= \frac{1}{2} \cdot (3 \cdot h_x^2 \cdot A_0 - A_1 \cdot h(x) + A_2) \cdot h_x \\ a_3(x) &= \frac{1}{2} \cdot h_x^2 \cdot \tilde{A}_1 + \tilde{A}_3, \end{aligned} \quad (2.10)$$

with some constants  $A_0, \tilde{A}_1, A_2, \tilde{A}_3$ . The notation in the formula above, especially  $\tilde{A}_1, \tilde{A}_3$ , is chosen for convenience in future formulas. Then the bracket yields

$$\{F, H\} = -h_x^2 \cdot (3 \cdot A_0 \cdot h_x^2 \cdot h_{xx} - A_1 \cdot h_x^2 - A_1 \cdot h_{xx} \cdot h(x) + A_2 \cdot h_{xx} - A_3) \cdot p_y^4.$$

This means that the equation on  $h(x)$  is

$$3 \cdot A_0 \cdot h_x^2 \cdot h_{xx} - A_1 \cdot h_x^2 - A_1 \cdot h_{xx} \cdot h(x) + h_{xx} \cdot A_2 = A_3. \quad (2.11)$$

The left hand side of this expression is the  $x$ -derivative of the expression

$$h_x \cdot (A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2). \quad (2.12)$$

Therefore the equation (2.11) is equivalent to

$$h_x \cdot (A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2) - (A_3 \cdot x + A_4) = 0. \quad (2.13)$$

**Remark 2.2.** Obviously, we obtain this equation from both the equations (1.2) (i) and (ii) taking the limit  $\mu \rightarrow 0$ . Moreover, the solution of the Cauchy initial value problem for the equations (2.8), (2.13) depends analytically on all parameters: the variable  $x$ , parameters  $\mu, A_0, \dots, A_4$ , the initial point  $x_0$ , and the initial value  $h(x_0)$ . Therefore we can consider real solutions  $h(x)$  of the equation (1.2) as “real forms” of a single holomorphic multi-valued function  $h(x; \mu; A_0, \dots, A_4; x_0, h_0)$  depending holomorphically on the involved parameters. Notice also that the equation (1.2) (ii) is obtained from (1.2) (i) by replacing  $\mu$  by  $i \cdot \mu$ , and similarly for the corresponding cubic integrals.

**Remark 2.3.** The above argumentation shows the existence of one non-trivial cubic integral  $F$  in the case  $\mu = 0$  (i.e., for the metric (2.3) with  $h$  satisfying (1.2)(iii)). Namely, such  $F$  can be obtained substituting the formulas (2.10) in (2.2). The solution  $F$  obtained in this way has the form  $F = \tilde{A}_3 \cdot L^3 + \tilde{A}_1 \cdot L \cdot H + F_1$  with a fixed cubic integral  $F_1$  which is linear in  $y$ . On the other hand, for  $\mu \neq 0$  in the both cases (i) and (ii) we obtain two non-trivial cubic integrals linearly independent of  $L^3$  and  $L \cdot H$ , namely, by replacing  $\mu$  by  $-\mu$  in formulas (2.5). It appears that also in the case  $\mu = 0$  there exists another cubic integral  $F_2$  that is (inhomogeneous) quadratic in  $y$ . The latter property is equivalent to the condition  $\mathcal{L}^3(F_2) = 0$ . We show the existence of such  $F_2$  in the proof of Theorem 5.1. This additional integral  $F_2$  is already included in the formulas in Theorem 1.1.

This fact is the reason for the difference in formulas (1.6) and (2.10). Namely, the substitution of (1.6) in (1.5) yields the linear combination  $C_1 F_1 + F_2 C_2$  of two cubic integrals  $F_1, F_2$  which are linear independent of  $L^3$  and  $L \cdot H$ . On the other hand, the substitution of (2.10) in (2.2) yields the linear combination  $F = \tilde{A}_3 \cdot L^3 + \tilde{A}_1 \cdot L \cdot H + F_1$  with the same  $F_1$ , which gives only one cubic integral linear independent of  $L^3$  and  $L \cdot H$ .

**2.4. Remaining steps of the proof.** As we have shown, if a surface metric  $g$  admits a linear and a non-trivial cubic integral  $F$ , then in appropriate coordinates it has the form  $h_x^{-2}(dx^2 + dy^2)$  for some function  $h(x)$  satisfying one of the forms (1.2) of the Principle equation, and that the cubic integral  $F$  can be constructed using the formula (2.5) or resp. (2.10). The remaining steps of the proof are the following:

- We analyse in which cases the constructed metric  $g = h_x^{-2}(dx^2 + dy^2)$  belongs to already known types: Metrics of constant curvature and Darboux-superintegrable metrics. This is done in Section 3. We show that our metrics are indeed new examples for most values of the parameters (the values of the parameters corresponding to previously known cases are solutions of certain algebraic equations).
- In Section 4 we prove that in the case of non-constant curvature the function  $h(x)$  satisfies a *unique up to constant factor* equation of type (1.2). This result is used to prove the fact that if the solution  $h(x)$  of the Principle equation with complex parameters  $\mu; A_0, \dots, A_4$  is real-valued, then the parameter  $\mu$  (which could be a priori arbitrary complex number) must be real, purely imaginary, or zero, whereas the parameters  $A_0, \dots, A_4$  becomes real after the application by the appropriate constant. This explains why we have only 3 types (i)–(iii) of the Principle equation (1.2).
- In Section 5 we prove that in the case of non-constant curvature every type (i)–(iii) of Theorem 1.1 the space of cubic integrals has dimension 4. This means that under hypotheses of the main theorem there are exactly 2 non-trivial independent cubic integrals, in addition to  $L^3$  and  $LH$ . This fact is a special case of Kruglikov’s “big

gap” conjecture (see [Kr]) about possible dimensions of the spaces of cubic integrals of surface metrics.

### 3. SPECIAL SOLUTIONS.

In this section we consider two special cases of the Principle equation corresponding to Darboux-superintegrable metrics and constant curvature metrics.

**3.1. The case  $A_0 = 0$  corresponds to Darboux-superintegrable metrics.** Recall that a two-dimensional metric  $g$  is *Darboux-superintegrable*, if the space of its quadratic integrals is at least 4-dimensional and the curvature is non-constant. We shall use the following statement which follows from [Kr] (or even from [Koe]): *if a metric  $g$  (with the Hamiltonian  $H$ ) of non-constant curvature admits a linear integral  $L$  and a quadratic integral  $Q$  such that  $L, Q$  and  $H$  are functionally independent, then  $g$  is Darboux-superintegrable.*

This statement implies that for every real solution  $h(x)$  of one of the equations(1.2) with  $A_0 = 0$  the metric  $g = h_x^{-2}(dx^2 + dy^2)$  is Darboux-superintegrable.

Indeed,  $A_0 = 0$  if and only if the integral  $F$  from Theorem 1.1 has zero coefficient at  $p_x^3$ . Since the linear integral  $L$  in Theorem 1.1 is  $p_y$ , the function  $Q := F/p_y$  is an integral quadratic in momenta. If  $L, H$  and  $F$  are functionally independent, then the functions  $L, H, Q$  are also functionally independent and the metric is Darboux-superintegrable by the result of [Koe, Kr] recalled above.

For further use let us note that every Darboux-superintegrable metric always has the form  $g = h_x^{-2}(dx^2 + dy^2)$  for some function  $h(x)$  satisfying one of the Principle equations (1.2) with  $A_0 = 0$ . Indeed, for given metric  $g$  admitting a non-vanishing linear integral  $L$  there exists a *isothermic* coordinate system  $(x, y)$ , unique up to translations, in which  $L = p_y$ . In these coordinates  $g$  has the form  $g = h_x^{-2}(dx^2 + dy^2)$  with *some* function  $h(x)$ . Further, if  $Q$  is a quadratic integral, then  $F := Q \cdot L$  is a cubic integral for  $g$ . In this situation we have shown that  $h(x)$  must satisfy one of the equations (1.2) with certain parameters  $A_0, \dots, A_4$  such that  $F = \sum_{i=0}^3 a_i(x, y) p_x^i p_y^{3-i}$  with  $a_0 = A_0 h_x^3$ . The condition  $F = Q \cdot L$  means the vanishing of  $a_0(x, y)$  which is equivalent to  $A_0 = 0$ .

### 3.2. Parameters in Theorem 1.1 corresponding to metrics of constant curvature.

The goal of this subsection is to understand for what values of the parameters  $A_0, \dots, A_4$  and the initial value  $h(x_0)$  the metric from Theorem 1.1 belong to the previously known classes, that is to the Darboux-integrable metrics and to the metrics of constant curvature. In §3.1 we have shown that Darboux-superintegrable metrics are characterized by the condition  $A_0 = 0$ . Thus in order to understand whether the metrics we constructed are new we need to understand which metrics with  $A_0 \neq 0$  have constant curvature. The answer is given in Theorem 3.1. In particular, Corollary 3.1 shows that most metrics we constructed are new.

Let  $g$  be a metric on  $M^2$  of the constant Gauss curvature  $R$  and  $\mathbf{v}$  a Killing vector field corresponding to the linear integral  $L_{\mathbf{v}}$ . Then according to the sign of  $R$  the Lie algebra of Killing vector fields on  $M^2$  is either  $\mathfrak{so}(3)$  (case  $R > 0$ ), or  $\mathfrak{sl}(2, \mathbb{R})$  (case  $R < 0$ ), or the affine algebra  $\mathfrak{aff}(\mathbb{R}^2)$  of isometries of  $\mathbb{R}^2$  isomorphic to a semi-direct sum  $\mathfrak{so}(2) \ltimes \mathbb{R}^2$  (remaining case  $R = 0$ ). The classification of elements of these three Lie algebras gives 6 types of Killing vector fields: rotations of  $S^2$  ( $R > 0$ ), rotations, hyperbolic translations, and loxodromies of the hyperbolic plane ( $R < 0$ ), and rotations and translations of  $\mathbb{R}^2$  ( $R = 0$ ). Fix a coordinate system  $(x, y)$  in which the metric has the form  $g = h_x^{-2}(dx^2 + dy^2)$ , the curvature is  $R = h_{xxx} \cdot h_x - h_{xx}^2$ , and the Killing vector field has the form  $\mathbf{v} = \frac{\partial}{\partial y}$ . Since in each of

these cases the metric has 3 Killing vector field, there exists a cubic integral independent of  $L_v^3$  and  $L_v \cdot H$ . Consequently,  $h(x)$  must satisfy one of the Principal equations. The explicit situation is as follows:

**Theorem 3.1.** *Assume that a metric  $g = h_x^{-2}(dx^2 + dy^2)$  has constant Gauss curvature  $R$ . Then  $h(x)$  satisfies one of the equations (1.2). Moreover, in this case under additional assumption  $A_0 \neq 0$  one of the following possibilities holds:*

- (1)  $h(x) = a \cdot \sinh(\mu \cdot (x - b)) + c$  with some constants  $\mu > 0, a > 0, c, b$  satisfying  $R = a^2 \mu^4$ . In this case the Killing vector field  $\frac{\partial}{\partial y}$  is locally a rotation of the 2-sphere of radius  $r = a^{-1} \mu^{-2}$  and Gauss curvature  $R = a^2 \mu^4$ . The function  $h(x)$  satisfies the equations (1.2) ii) (elliptic type) in the form

$$\begin{aligned} h_x \cdot (h_x^2 - \mu^2(h(x) - c)^2 + C) &= a\mu \cdot (C + (a\mu)^2) \cdot \cosh(\mu \cdot (x - b)) \\ h_x \cdot (h_x^2 - (3\mu)^2(h(x) - c)^2 - 3 \cdot (\mu a)^2) &= -2 \cdot (a\mu)^3 \cdot \cosh(3\mu \cdot (x - b)) \end{aligned} \quad (3.1)$$

with arbitrary constant  $C$  in the first equation;

- (2)  $h(x) = a \cdot \cosh(\mu \cdot (x - b)) + c$  with some constants  $\mu > 0, a > 0, c, b$  satisfying  $R = -a^2 \mu^4$ . In this case the Killing vector field  $\frac{\partial}{\partial y}$  is locally a rotation of the hyperbolic plane of constant Gauss curvature  $R = -a^2 \mu^4$  and  $h(x)$  satisfies the equation (1.2) ii) (elliptic type) in the form

$$\begin{aligned} h_x \cdot (h_x^2 - \mu^2(h(x) - c)^2 + C) &= a\mu \cdot (C - (a\mu)^2) \cdot \sinh(\mu \cdot (x - b)) \\ h_x \cdot (h_x^2 - (3\mu)^2(h(x) - c)^2 + 3 \cdot (\mu a)^2) &= -2 \cdot (a\mu)^3 \cdot \sinh(3\mu \cdot (x - b)) \end{aligned} \quad (3.2)$$

with arbitrary constant  $C$  in the first equation;

- (3)  $h(x) = a \cdot \sin(\mu \cdot (x - b)) + c$  with some constants  $\mu > 0, a > 0, c, b$  satisfying  $R = -a^2 \mu^4$ . In this case the Killing vector field  $\frac{\partial}{\partial y}$  is locally a translation on the hyperbolic plane of constant Gauss curvature  $R = -a^2 \mu^4$  and  $h(x)$  satisfies the equation (1.2) i) (hyperbolic type) in the form

$$\begin{aligned} h_x \cdot (h_x^2 + \mu^2(h(x) - c)^2 + C) &= a\mu \cdot (C + (a\mu)^2) \cdot \cos(\mu \cdot (x - b)) \\ h_x \cdot (h_x^2 + (3\mu)^2(h(x) - c)^2 - 3 \cdot (\mu a)^2) &= 2 \cdot (a\mu)^3 \cdot \cos(3\mu \cdot (x - b)) \end{aligned} \quad (3.3)$$

with arbitrary constant  $C$  in the first equation;

- (4)  $h(x) = a \cdot (x - b)^2 + c$  with some constants  $a > 0, c, b$  satisfying  $R = -4a^2$ . In this case the Killing vector field  $\frac{\partial}{\partial y}$  is a loxodromy on the hyperbolic plane of constant Gauss curvature  $R = -4a^2$  and  $h(x)$  satisfies the equation (1.2) iii) (parabolic/nilpotent type)

$$h_x \cdot (h_x^2 - 4 \cdot a \cdot h(x) + A_2) = 2a \cdot (A_2 - 4 \cdot a \cdot c) \cdot (x - b) \quad (3.4)$$

with arbitrary constant  $A_2$ ;

- (5)  $h(x) = a \cdot \exp(\mu x) + c$  with some constants  $\mu > 0, a > 0, c$  and  $R = 0$ . In this case the Killing vector field  $\frac{\partial}{\partial y}$  is locally a rotation of the Euclidean plane ( $R = 0$ ) and  $h(x)$  satisfies the equation (1.2) i) (hyperbolic type) in the form

$$\begin{aligned} h_x \cdot (h_x^2 - \mu^2(h(x) - c)^2 + C) &= -a\mu C \cdot \exp(\mu x) \\ h_x \cdot (h_x^2 - (3\mu)^2(h(x) - c)^2) &= -8 \cdot (a\mu)^3 \cdot \exp(3\mu x) \end{aligned} \quad (3.5)$$

with arbitrary constant  $C$  in the first equation;

(6)  $h(x) = a \cdot x + c$  with some constants  $a > 0, c$ . In this case  $R = 0$ , the Killing vector field  $\frac{\partial}{\partial y}$  is locally a translation of the Euclidean plane ( $R = 0$ ), and  $h(x)$  satisfies the equation (1.2) iii) (parabolic/nilpotent type)

$$h_x \cdot (h_x^2 - A_1 \cdot h(x) + A_2) = -a^2 A_1 \cdot x + a \cdot (a^2 - c \cdot A_1 + A_2) \quad (3.6)$$

with arbitrary constants  $A_1, A_2$ .

*Proof.* As we have shown above, if a metric  $g$  admits a Killing vector field  $\mathbf{v}$ , then in appropriate coordinates  $g$  has the form  $g = h_x^{-2}(dx^2 + dy^2)$  and the Killing vector field the form  $\mathbf{v} = \frac{\partial}{\partial y}$ . In this case the Gauss curvature  $R$  is given by  $R = h_{xxx} \cdot h_x - h_{xx}^2$ . Thus we are interesting in possible solutions of the ODE  $h_{xxx} \cdot h_x - h_{xx}^2 = R$  with constant parameter  $R$  such that  $h_x \neq 0$ . By direct calculations we see that every function on the list items (1)–(6) satisfies the ODE  $h_{xxx} \cdot h_x - h_{xx}^2 = R$  with an appropriate constant  $R$ , and the theorem claims that the list is complete. In view of the uniqueness of the solution of an ODE with the given initial values we must show that every combination of the initial values  $I := (R, x_0, h(x_0), h_x(x_0), h_{xx}(x_0))$  is realised by one of the solution on the list. Inverting the sign of  $h(x)$  and  $x$ , if needed, we may assume that  $h_x(x_0) > 0$  and  $h_{xx}(x_0) > 0$ .

Let us consider  $h_{xxx}(x_0) = \frac{R + h_{xx}(x_0)^2}{h_x(x_0)}$ . If  $h_{xxx}(x_0) = 0$  then the data  $I$  are realised by an appropriate polynomial of degree 2 (case  $h_{xx}(x_0) \neq 0$ , list item (4)) or 1 (case  $h_{xx}(x_0) = 0$ , list item (6)).

In the case  $h_{xxx}(x_0) < 0$  we set  $\mu := \sqrt{-h_{xxx}(x_0)/h_x(x_0)}$  and  $a := \sqrt{\mu^{-2}h_x^2(x_0) + \mu^{-4}h_{xx}^2(x_0)}$ . It is not difficult to see that the ODE  $h_{xxx} \cdot h_x - h_{xx}^2 = R$  admits the solution  $h(x) = a \cdot \sin(\mu \cdot (x - b)) + c$  (list item (3)) with appropriate parameters  $b$  and  $c$  satisfying the initial conditions  $I$ .

In the remaining case  $h_{xxx}(x_0) > 0$  we set  $\mu := \sqrt{h_{xxx}(x_0)/h_x(x_0)}$  and look for the solution of the equation  $h_{xxx} \cdot h_x - h_{xx}^2 = R$  in one of the forms (1), (2), or (5) with appropriate parameters  $a > 0, b, c$ . The form (1) is realised in the case  $h_{xx}(x_0) < \mu \cdot h_x(x_0)$  in which  $R > 0$ , the form (2) in the case  $h_{xx}(x_0) > \mu \cdot h_x(x_0)$  in which  $R > 0$ , and form (5) in the case  $h_{xx}(x_0) = \mu \cdot h_x(x_0)$  in which  $R = 0$ . The needed parameters  $a > 0, b, c$  can be found easily.

It remains to show every function  $h(x)$  given by one of the formulas (1)–(6) satisfies one of the ODEs (1.2) with  $A_0 \neq 0$  and determine possible values of the parameters  $\mu$  and  $A_0, \dots, A_4$ . Due to the condition  $A_0 \neq 0$  we may assume that  $A_0 = 1$ . The key observation is that  $h(x)$  is (up to a constant) either a trigonometric (case (3)), or trig-hyperbolic (cases (1) and (2)), or exponential (case (5)), or a usual monomial (cases (4) and (6)) and therefore the differential expression  $h_x \cdot (h_x^2 \pm \tilde{\mu}^2 \cdot h^2(x) - A_1 \cdot h(x) + A_2)$  will be a polynomial of the same type, divisible by the monomial  $h_x$ , for example,  $\sum_j B_j e^{j\mu x}$  in the exponential case.

In the cases (1) and (2) we conclude that the right hand side must be of the form  $B_1 \cdot \cosh(k\mu) + B_2 \cdot \sinh(k\mu)$  with  $k = 1, 2$  or  $3$  which gives  $\tilde{\mu} = k\mu$ . The case  $k = 2$  is excluded by the argument that for  $k \neq \pm 1$  the expression  $h_x \cdot (h_x^2 \pm \tilde{\mu}^2 \cdot h^2(x)) = h_x \cdot (h_x^2 \pm k^2 \mu^2 \cdot h^2(x))$  is a trig-hyperbolic polynomial of degree 3, i.e., containing a term  $\cosh(3\mu)$  or a term  $\sinh(3\mu)$ . Using the relations  $\sinh(3x) = 4 \cdot \sinh^3(x) + 3 \cdot \sinh(x)$  and  $\cosh(3x) = 4 \cdot \cosh^3(x) - 3 \cdot \cosh(x)$  we conclude that the only possible equations are (3.1) and (3.2).

The remaining case (3)–(6) involving trigonometric polynomials, exponential polynomials, and usual polynomials instead of trig-hyperbolic ones are treated in the same manner.  $\square$

**Corollary 3.1.** *Every equation (1.2) with  $A_0 \neq 0$  and  $(A_3, A_4) \neq (0, 0)$  admits only finitely many (real) solutions  $h(x)$  such that the metric  $g = h_x^{-2}(dx^2 + dy^2)$  has constant Gauss curvature, except the case of the equation  $h_x(A_0 \cdot h_x^2 + A_2) = A_4$  which always admits a solution of the form  $h(x) = a \cdot x + c$  with arbitrary  $c$  and  $a$  satisfying  $a(A_0 \cdot a^2 + A_2) = A_4$ .*

Every real solution  $h(x)$  of (1.2) is completely determined by its initial values  $h(x_0), h_x(x_0)$  at a given point  $x_0$ . Thus for a generic choice of the initial value  $h(x_0)$  the solution  $h(x)$  of the equation (1.2) with this initial value and with any root  $h_x(x_0)$  of the corresponding algebraic equation at  $x_0$  the metric  $g = h_x^{-2}(dx^2 + dy^2)$  has non-constant Gauss curvature.

*Proof.* As we have seen, a metric of the form  $g = h_x^{-2}(dx^2 + dy^2)$  has constant curvature if and only if  $h(x)$  is one of the forms (1)–(6). Let us consider possible right hand sides.

Every expression  $A_3 \cdot \frac{\sin(\mu \cdot x)}{\mu} + A_4 \cdot \cos(\mu \cdot x)$  can be written in the form  $A \cdot \cos(\mu \cdot (x - b))$  with unique  $A$  and  $b$  unique up to a multiple of the period. Similarly, every expression  $A_3 \cdot \frac{\sinh(\mu \cdot x)}{\mu} + A_4 \cdot \cosh(\mu \cdot x)$  can be uniquely written in one of the following forms:  $A \cdot \cosh(\mu \cdot (x - b))$ ,  $A \cdot \sinh(\mu \cdot (x - b))$ ,  $A \cdot \exp(\mu \cdot x)$ , or  $A \cdot \exp(-\mu \cdot x)$ . The latter case can be reduced to the previous one by inverting the  $x$ -axis. Thus the right hand side of the equation (1.2), ii), determines which type (1), (2), or (5) of the solution  $h(x)$  we obtain, and in the case (1.2), i), the solution must be of the type (3).

In the case when  $h(x)$  is a solution of the type (1), (2), or (3) we proceed as follows: Comparing the right hand side of equations (1.2) and (3.1)–(3.3) we determine  $b$  and possible values of  $\mu$ , there are only finitely many such possibilities. Then multiplying the equation by a constant we make  $A_0 = 1$ . Next, we compare the l.h.s. and determines the parameters  $c$  and  $C$ . After this the right hand side of (3.1)–(3.3) determines the possible values of  $a$ . Clearly, we have only finitely many possibilities.

Notice that the type (5) is not generic itself since it occurs only if  $A_3 = \pm \mu \cdot A_4$ . Nevertheless, in this case for a given  $A_0, \dots, A_4$  we still have only finitely many solutions  $h(x)$  giving constant curvature. Indeed, we determine possible values of  $\mu$  considering the right hand side of the equation, then from l.h.s. we determine possible values of the parameters  $c$  and  $C$ , and finally again from the right hand side we determine  $a$ .

In the case (4) when  $h(x) = a \cdot (x - b)^2 + c$  we must have  $b = -A_4/A_3$  and  $a = A_1/4$ , and finally  $c = \frac{2aA_2 - A_3}{8a^2}$ . So for given  $A_0, \dots, A_4$  we could have at most one solution of type (4).

Finally, if  $A_1 \neq 0$  and  $h(x)$  is of type (6), i.e.  $h(x) = a \cdot x + c$ , then  $a$  must satisfy  $A_3 = -a^2 \cdot A_1$  which gives us at most two possibilities. For every  $a$  we have the unique possibility for  $c$ .  $\square$

#### 4. UNIQUENESS OF THE PRINCIPLE EQUATION

The uniqueness of the Principle equation is an interesting phenomenon *per se* and plays an important role in the proof of the main theorem. We shall need the following two results.

**Lemma 4.1.** *Let  $h(x)$  be a complex-valued solution of the equation*

$$\mathcal{E} := h_x(h_x^2 - 9 \cdot h(x)^2 + A_2) - A_+ e^{3x} - A_- e^{-3x} = 0 \quad (4.1)$$

*with complex coefficients  $A_+ \neq 0, A_-, A_2$  defined for  $x \in [x^*, +\infty)$ . Assume that for  $x \rightarrow +\infty$  the function  $h(x)$  has the asymptotic growth  $h(x) = a \cdot e^x + o(e^x)$  with  $a \neq 0$ . Then  $A_+ = -8a^3$  and there exists a complex-valued real-analytic function  $f(\tau)$  defined for sufficiently small  $\tau$  such that  $f(0) = a$  and  $h(x) = e^x \cdot f(e^{-2x})$ .*

*Proof.* Write  $h_x(x) = \psi(x) \cdot e^x$ , substitute this expression in (4.1), and consider the obtained relation as a cubic algebraic equation on a variable  $\psi$  depending on the parameter  $x$ . Then for  $x \rightarrow +\infty$  (the coefficients of) the obtained equation converge to  $\psi(\psi^2 - 9a^2) = A_+$ . This implies the asymptotic growth  $h_x(x) = a' \cdot e^x + o(e^x)$  with some  $a'$  satisfying the equation  $a'(a'^2 - 9a^2) = A_+$ . Integrating it we obtain the asymptotic  $h(x) = a' \cdot e^x + o(e^x)$ . Consequently,  $a' = a$  and hence  $a$  satisfies  $A_+ = -8a^3$ .

Now make the substitution  $x = -\frac{1}{2} \log(\tau)$  and  $h(x) = e^x \cdot (a + a_0 e^{-2x} + e^{-4x} f(e^{-2x})) = \tau^{-1/2}(a + a_0 \tau + \tau^2 f(\tau))$ . Then the equation (4.1) transforms into

$$\begin{aligned} & 8f_\tau^3 \tau^8 + 36ff_\tau^2 \tau^7 + 12f_\tau(3f^2 + f_\tau a_0) \tau^6 - 12af_\tau^2 \tau^5 \\ & -12(3a_0 f^2 + 6ff_\tau a + a_0^2 f_\tau) \tau^4 + (2A_2 f_\tau - 48aa_0 f_\tau - 72af^2 - 36a_0^2 f) \tau^3 \\ & + (3A_2 f + A_- - 72aa_0 f - 8a_0^3 - 12a^2 f_\tau) \tau^2 \\ & + (A_2 - 12aa_0)(a_0 \tau - a) = 0. \end{aligned} \quad (4.2)$$

This means that we are now looking for solutions  $f(\tau)$  of (4.2) defined for small  $\tau > 0$ . The condition on the growth of  $h(x)$  and  $h_x$  means that  $f(\tau) = o(\tau^{-3/2})$  and  $f_\tau(\tau) = o(\tau^{-2})$ . Therefore we must have  $a_0 = \frac{A_2}{12a}$  and the substitution  $A_2 = 12aa_0$  transforms the equation (4.2) into

$$\begin{aligned} & 8f_\tau^3 \tau^6 + 36ff_\tau^2 \tau^5 + 12f_\tau(3f^2 + f_\tau a_0) \tau^4 - 12af_\tau^2 \tau^3 \\ & -12(3a_0 f^2 + 6ff_\tau a + a_0^2 f_\tau) \tau^2 - 12(2aa_0 f_\tau + 6af^2 + 3a_0^2 f) \tau \\ & + (A_- - 12a^2 f_\tau - 36aa_0 f - 8a_0^3) = 0. \end{aligned} \quad (4.3)$$

For any given  $A_-, a, a_0, f$  and sufficiently small  $\tau$  the latter relation can be resolved in  $f_\tau$  as a real-analytic function  $f_\tau = F(\tau, f, a, a_0, A_-)$  with  $F(\tau, f, a, a_0, A_-) = \frac{A_- - 36aa_0 f(\tau) - 8a_0^3}{12a^2} + O(\tau)$ .

Consequently, we can conclude the following properties: Any solution  $h(x)$  of (4.1) satisfying the hypotheses of the lemma is given by the series  $h(x) = ae^x + \sum_{k=0}^{\infty} a_k e^{-(2k+1)x}$  which converges for  $x \in [x_0, +\infty)$ . Moreover, the coefficients  $a, a_0$  satisfy the relations  $8a^3 = -A_+$  and  $12aa_0 = A_2$ .

Furthermore, we can also conclude the following two existence results for solutions of (4.1):

- First, for  $a, a_0$  satisfying the conditions above, for any given sufficiently large  $x_0 \gg 0$  and any sufficiently small  $b \in \mathbb{C}$  there exists a unique solution of (4.1) with the initial value  $h(x_0) = a^{x_0} + a_0 e^{-x_0} + b e^{-3x_0}$ .

- Second, for  $a, a_0$  satisfying the conditions above and any given  $a_1$  there exists a unique solution of (4.1) which is defined for  $x \gg 0$  and whose initial terms in the series above have coefficients  $a, a_0, a_1$ .  $\square$

**Lemma 4.2.** *Assume that a complex-valued function  $h(x)$  satisfies the equation*

$$h_x(A_0(h_x^2 - \mu^2 h(x)^2) - A_1 h(x) + A_2) = 0.$$

*with complex parameters  $\mu, A_0, A_1, A_2$ . Then  $R := h_{xxx} h_x - h_{xx}^2$  is constant.*

*Proof.* We have obviously two possibilities: Either  $h_x$  vanishes identically,  $h_x \equiv 0$ , or

$$A_0(h_x^2 - \mu^2 h(x)^2) - A_1 h(x) + A_2 \equiv 0. \quad (4.4)$$

The first case  $h_x \equiv 0$  is trivial since then  $R = h_{xxx} h_x - h_{xx}^2 \equiv 0$ . So we may assume that  $h_x$  is not vanishing identically.

Assume additionally that  $\mu \neq 0$ . In the case  $A_0 = 0$  the solution of (4.4) is a constant function, and  $R$  vanishes also. In the case  $A_1 = A_2 = 0$  the solution of (4.4) is  $h(x) = C \cdot \exp(\pm \mu x)$ , and again  $R$  vanishes. In the remaining case  $A_0 \neq 0 \neq A_1$  every solution of (4.4) has the form  $h(x) = c_0 + c_1 \sinh(\pm \mu x + c_2)$  with arbitrary  $c_2$  and appropriate  $c_0, c_1$ . This time  $R$  must be constant too.

In the case  $\mu = 0$  the argumentation is changed as follows. If  $A_0 = 0$ , then  $h(x)$  must be constant, and then  $R \equiv 0$ . If  $A_1 = 0$ , then  $h(x)$  must be linear, which is also a contradiction. Finally, in the case  $A_0 \neq 0 \neq A_1$  every solution  $h(x)$  of (4.4) with  $\mu = 0$  is quadratic in  $x$ , and then  $R := h_{xxx} \cdot h_x - h_{xx}^2$  is constant again.  $\square$

**Theorem 4.1.** *Let  $h(x)$  be a complex function defined in a some open set  $U \subset \mathbb{C}$  satisfies two equations each of the form (1.2) with some complex parameters  $A_0, \dots, A_4, \mu$  and respectively  $B_0, \dots, B_4, \lambda$ . Assume that  $R := h_{xxx} \cdot h_x - h_{xx}^2$  is not constant. Then  $\mu = \pm \lambda$  and the equations are proportional.*

**Remark 4.1.** For convenience in calculation below we consider only equations of the form (1.2) (ii) or (iii), but not (1.2) (i). This is an equivalent problem, since the substitution  $\mu \mapsto i\mu$  switches between forms (1.2) (i) and (ii).

*Proof.* One of the techniques to prove the theorem is to write a Taylor series  $h(x) = \sum_j a_j (x - x_0)^j$ , substitute it in both equations, write the expansions, and then compare term by term the coefficients. In some places we use another approach, namely, we study geometric properties of the solution  $h(x)$  using methods of geometric function theory and algebraic geometry. It should be noticed however that every relation which will be obtained by geometric methods can be also received purely algebraically from the equations obtained from Taylor series.

Denote by  $\mathcal{E}_\lambda$  and  $\mathcal{E}_\mu$  the equations from the hypotheses of the theorem, and  $\Lambda_\mu(x), \Lambda_\lambda(x)$  the r.h.s.-s of these equations. In the case  $\mu \neq 0$  and resp.  $\lambda \neq 0$  we rewrite them as  $\Lambda_\mu = A_+ e^{\mu x} + A_- e^{-\mu x}$  and resp.  $\Lambda_\lambda = B_+ e^{\lambda x} + B_- e^{-\lambda x}$ . Notice that by Lemma 4.2  $\Lambda_\lambda(x)$  and  $\Lambda_\mu$  are non-zero.

We claim that the function  $h(x)$  extends to an analytic multi-sheeted ( $\Leftrightarrow$  multi-valued) function of  $x \in \mathbb{C} \setminus S$  for some discrete set  $S \subset \mathbb{C}$ . Let us consider several cases. The first is when  $A_0 \neq 0$ . In this case the equation  $\mathcal{E}_\mu$  is a polynomial of degree 3 in  $h_x$ . Let  $\mathcal{D}_\mu(x)$  be its discriminant with respect to  $h_x$ . Then  $\mathcal{D}_\mu(x) = -27\Lambda_\mu^2 - 4(A_0\mu^2 h(x)^2 - A_1 h(x) + A_2)^3$ , this is an analytic function in  $x$  defined in the domain of definition  $U$  of  $h(x)$ . In the case when  $\mathcal{D}_\mu(x)$  is not identically zero we can resolve the equation  $\mathcal{E}_\mu$  as an analytic multi-sheeted ( $\Leftrightarrow$  multi-valued) function of  $x \in U \setminus S$  where  $S$  is the set of zeroes of  $\mathcal{D}_\mu(x)$ . This transforms the equation in the explicit form  $h_x = F(x, h(x))$  where  $F$  is multi-sheeted function with ramifications exactly at zero points of the discriminant  $\mathcal{D}_\mu$ . This gives us the claim.

In the case when the discriminant  $\mathcal{D}_\mu$  vanishes identically  $h_x$  satisfies also the equation  $3h_x^2 + A_0\mu^2 h(x)^2 - A_1 h(x) + A_2 = 0$  which is the derivative of  $\mathcal{E}_\mu$  with respect to  $h_x$ . Again we obtain an explicit equation  $h_x = F(x, h(x))$  with analytic multi-sheeted right hand side  $F(x, h(x))$  with singularities in some discrete subset  $S$ . Hence this time also  $h(x)$  extends to an analytic multi-sheeted function of  $x \in \mathbb{C} \setminus S$  for some discrete set  $S \subset \mathbb{C}$ . Finally, in the case  $A_0 = 0$  the equation  $\mathcal{E}_\mu$  can be resolved as  $h_x = \frac{\Lambda_\mu(x)}{A_2 - A_1 h(x)}$ , and we can conclude the claim.

Assume that  $A_0 \neq 0 \neq B_0$ . Then dividing equations by  $A_0$  or resp.  $B_0$  we reduce the general situation to the case  $A_0 = B_0 = 1$ .

First, we prove that under the hypotheses of the theorem we must have the relation  $\lambda = \pm\mu$ . Let us assume the contrary, i.e.,  $\lambda \neq \pm\mu$ . We shall consider numerous special cases and subcases.

Without loss of generality we may suppose that  $|\lambda| \geq |\mu|$ . In particular,  $\lambda \neq 0$ . Observe that under complex affine transformation  $x \mapsto ax + b$  the equations  $\mathcal{E}_\lambda, \mathcal{E}_\mu$  retain their structure only changing the parameters  $\mu, \lambda, A_i, B_i$ , in particular,  $\lambda$  transforms in  $\frac{\lambda}{a}$ . Consequently, we may assume that  $\lambda$  is real positive,  $\lambda > 0$ . Then by assumption on  $\mu$  we conclude  $|\Re(\mu)| < \lambda$ . Further, recall that  $\Lambda_\lambda(x) = B_+ \cdot \exp(\lambda x) + B_- \cdot \exp(-\lambda x)$  such that at least one constant  $B_+, B_-$  is non-zero. Inverting the coordinate  $x$ , if needed, we can suppose that  $B_+ \neq 0$ .

Assume additionally that  $\mu \neq 0$ . Then  $\Lambda_\mu(x) = A_+ \cdot \exp(\mu x) + A_- \cdot \exp(-\mu x)$ . Consider the difference  $\mathcal{E}_\lambda - \mathcal{E}_\mu$ . It has the form

$$h_x \cdot (C_0 h(x)^2 + C_1 h(x) + C_2) = \Lambda_\delta(x) \quad (4.5)$$

where  $C_0 = \lambda^2 - \mu^2$  and  $\Lambda_\delta(x) = \Lambda_\lambda(x) - \Lambda_\mu(x)$ . Denote this equation by  $\mathcal{E}_\delta$ . Integrating it, we obtain an *algebraic* equation

$$\frac{1}{3} C_0 h(x)^3 + \frac{1}{2} C_1 h(x)^2 + C_2 h(x) = \tilde{\Lambda}_\delta(x) + C_3 \quad (4.6)$$

which we denote by  $\tilde{\mathcal{E}}_\delta$  and in which  $C_3$  is some constant and  $\tilde{\Lambda}_\delta(x) = \int \Lambda_\delta(x) dx$  equals

$$\tilde{\Lambda}_\delta(x) = \frac{B_+}{\lambda} \cdot \exp(\lambda x) - \frac{B_-}{\lambda} \cdot \exp(-\lambda x) + \frac{A_+}{\mu} \cdot \exp(\mu x) - \frac{A_-}{\mu} \cdot \exp(-\mu x). \quad (4.7)$$

Making a translation in  $x$ , we can suppose that the real axis  $x \in \mathbb{R}$  does not contain singular points of  $h(x)$ , and that our open set  $U \subset \mathbb{C}$  hits the real axis  $x \in \mathbb{R}$ . Then there exists a unique extension of the function  $h(x)$  over the axis  $x \in \mathbb{R}$  which satisfies the equation (4.6).

Since  $C_0 = \lambda^2 - \mu^2 \neq 0$ , the function  $h(x)$  has asymptotic expansion  $h(x) \sim \left(\frac{3B_+}{C_0\lambda}\right)^{1/3} \exp(\lambda x/3)$  for  $x \rightarrow +\infty$ . From (4.5) we conclude that the derivative  $h_x$  has asymptotic expansion  $h_x \sim \left(\frac{3B_+}{C_0\lambda}\right)^{1/3} \cdot \frac{\lambda}{3} \cdot \exp(\lambda x/3)$  for  $x \rightarrow +\infty$ .

Further, assume that  $\lambda \neq \pm 3\mu$ . Then  $h_x^2 - \mu^2 h(x)^2$  has asymptotic expansion  $\sim C \cdot \exp(2\lambda x/3)$  for  $x \rightarrow +\infty$ , which gives the asymptotic expansion  $\sim C \cdot \exp(\lambda x)$  for the r.h.s. of  $\mathcal{E}_\mu$ . But by our condition  $|\Re(\mu)| < \lambda$  the l.h.s.  $\Lambda_\mu(x)$  has slower growth. The obtained contradiction gives the proof in the case  $\lambda \neq \pm 3\mu$ .

It remains to consider the case  $\lambda = \pm 3\mu$ . However, before this case we notice that the consideration above are valid also in the case  $\mu = 0 \neq \lambda$ . Indeed, we obtain the same growth asymptotic  $h(x) \sim \left(\frac{3B_+}{C_0\lambda}\right)^{1/3} \cdot \frac{\lambda}{3} \cdot \exp(\lambda x/3)$  for  $x \rightarrow +\infty$ , and the same contradiction in growth of right and left hand sides of  $\mathcal{E}_\mu$ .

Now we consider the case  $\lambda = \pm 3\mu$ . Recall that we assume that  $A_0 \neq 0 \neq B_0$ . Changing the sign of  $\mu$ , if needed, we obtain  $\lambda = +3\mu$ . Rescaling  $x$ , we can make  $\lambda = 3$  and  $\mu = 1$ . Adding to  $h(x)$  a constant we may assume that  $B_1 = 0$ . At this point we apply Lemma 4.1 to  $h(x)$  and the equation  $\mathcal{E}_\lambda$ . It gives us the presentation of  $h(x)$  as a series  $h(x) = ae^x + a_0 e^{-x} + a_1 e^{-3x} + \dots$  which converges for  $x \gg 0$  such that  $B_+ = -8a^3$  and  $12aa_0 = B_2$ . Differentiating the series we obtain  $h_x = ae^x - a_0 e^{-x} - 3a_1 e^{-3x} + \dots$ . This gives us  $h_x^2 - h(x)^2 = (h_x - h(x)) \cdot (h_x + h(x)) = -4aa_0 - 8aa_1 e^{-2x} + \dots$ . In the case  $A_1 \neq 0$  the

growth of the l.h.s. of  $\mathcal{E}_\mu$  is  $\sim e^{2x}$ , which is faster than in the r.h.s.  $\Lambda_\mu \sim e^x$ . Consequently,  $A_1 = 0$ .

Consider the case  $B_- \neq 0$ . Then repeating the argumentation above, we obtain the asymptotic behaviour  $h(x) \sim a_- e^{-x}$  for  $x \rightarrow -\infty$ . After this we can apply Lemma 4.1 to  $h(x)$  and the equation  $\mathcal{E}_\lambda$  in the negative range  $x \in (-\infty, x_0]$ , yielding a similar expansion  $h(x) = a_- e^{-x} + a_{0,-} e^x + a_{1,-} e^{3x} + \dots$ , in which  $B_- = 8a_-^3$  and  $12a_- a_{0,-} = B_2$ . Under translation  $x \mapsto x + \xi$  with  $\xi \in \mathbb{C}$  the coefficients  $a, a_-$  transform as  $a \mapsto a e^{-\xi}, a_- \mapsto a_- e^{+\xi}$ . Consequently after an appropriate translation we achieve the equality  $a = -a_-$ , and hence the equality  $B_+ = -8a^3 = 8a_-^3 = B_-$ . The relations  $B_2 = 12aa_0 = 12a_- a_{0,-}$  gives also the equality  $a_{0,-} = -a_0$ .

The substitution of the series in  $\mathcal{E}_\mu$  yields  $(aA_2 - A_+ - 4a^2 a_0)e^x + O(e^{-x})$  for  $x \rightarrow +\infty$  and  $(-a_- A_2 - A_- + 4a_-^2 a_{0,-})e^{-x} + O(e^x)$  for  $x \rightarrow -\infty$ . Since  $\mathcal{E}_\mu$  vanishes identically, we obtain  $A_+ = aA_2 - 4a^2 a_0 = -a_- A_2 - 4a_-^2 a_{0,-} = A_-$ . The next term in the expansion of  $\mathcal{E}_\mu$  for  $x \rightarrow +\infty$  is

$$(-8a^2 a_1 + 4a_0^2 a - a_0 A_2 + 4a^2 a_0 - a A_2)e^{-x}.$$

Since it must vanish, we obtain  $a_1 = -(aA_2 - 4a_0^2 a + a_0 A_2 - 4a^2 a_0)/(8a^2)$ . Making the same computation for  $x \rightarrow -\infty$  we obtain  $a_{1,-} = -a_1$ .

We conclude that the solution  $h(x)$  has sheets which satisfy the relation  $h(-x) = -h(x)$ . Further, recall that  $h(x)$  satisfies the algebraic equation  $\tilde{\mathcal{E}}_\delta$ , see (4.6). Since  $A_+ = A_-$  and  $B_+ = B_-$ ,  $\tilde{\Lambda}_\delta$  (given by (4.7) with  $\lambda = 3$  and  $\mu = 1$ ) is an odd function,  $\tilde{\Lambda}_\delta(-x) = -\tilde{\Lambda}_\delta(x)$ . Consequently,  $C_3$  vanishes.

Let  $w(z)$  be the 3-sheeted function of the argument  $z \in \mathbb{C}$  given by the algebraic equation  $\frac{8}{3}w^3 + C_2 w = z$  with  $C_2 \neq 0$ . Then  $h(z) = w(\tilde{\Lambda}_\delta(x))$ . Let us observe the following facts about the function  $w(z)$ : The first is that for  $z$  small enough three branches of  $w(z)$  are given by the approximate formulas  $w_0(z) = z/C_2 + O(z^2)$ ,  $w_+(z) = +\sqrt{-3C_2/8}$ , and  $w_-(z) = -\sqrt{-3C_2/8}$ . The second is that the monodromy group of  $w(z)$  is the symmetric group permuting the sheets of  $w(z)$ . In particular going along an appropriate path in  $z$ -plane we can interchange two given sheets of  $w(z)$ .

In turn, the function  $z = \tilde{\Lambda}_\delta(x)$  is the composition of the polynomial  $Q(u) := \frac{8}{3}B_+ u^3 + 2(B_+ - A_+)u$  with the functions  $u = \sinh(x)$ . The polynomial  $Q(u)$  has the same structure  $c_0 z^3 + c_2 z$  as the polynomial  $\frac{8}{3}w^3 + C_2 w$ . Consequently, there are two possibilities: Either the critical values of the polynomials  $Q(u)$  and  $\frac{8}{3}w^3 + C_2 w$  are different, and then monodromy of the composition  $w(Q(u))$  in the full symmetric group as in the case of the function  $w(z)$ , or the difference  $\frac{8}{3}w^3 + C_2 w - Q(u)$  splits in the product  $\frac{8}{3} \prod_{j=1}^3 (w - b_j u)$  with appropriate  $b_j \in \mathbb{C}$ . In the latter case three possible branches of  $h(x)$  are  $w(u(\sinh(x))) = b_j \sinh(x)$ , and in this case  $R = h_{xxx} h_x - h_{xx}^2$  is constant in contradiction with the assumption of the theorem. Consequently, the latter case is impossible, and the monodromy of the composition  $w(Q(u))$  is the full symmetric group.

Finally, the critical points of the function  $u = \sinh(x)$  are given by the condition  $\sinh'(x) = \cosh(x) = 0$ , and hence the critical values of  $u = \sinh(x)$  are  $\pm i$ . The corresponding values of  $z = Q(u(x))$  are  $z_\pm := \pm i(\frac{2}{3}B_+ + 2A_+)$ . Here we notice that for each value  $z_\pm$  has at least two pre-images  $Q^{-1}(z_\pm)$  such that at most one of them is  $\pm i$ , and neither of these pre-images is 0. Further, we notice that the function  $u = \sinh(x)$  is surjective. Indeed, it is the composition of the rational function  $u(t) = \frac{1}{2}(t - t^{-1})$  and the exponent  $t = \exp(x)$ , the map  $u(t) = \frac{1}{2}(t - t^{-1})$  acts surjectively from  $\mathbb{C} \setminus \{0\}$  onto  $\mathbb{C}$ , the map  $t = \exp(x)$  surjectively from  $\mathbb{C}$  onto  $\mathbb{C} \setminus \{0\}$ . Summing up we conclude that for any of two critical values  $z^*$  of the

polynomial  $\frac{8}{3}w^3 + C_2w$  there exists an  $x^* \in \mathbb{C}$  which is not a critical point of the function  $z = Q(\sinh(x))$  but  $z^* = Q(\sinh(x^*))$ . This shows that the monodromy of the function  $h(x)$  is the full symmetric group permuting its 3 sheets ( $\Leftrightarrow$  branches).

In particular, each of 3 sheets of  $h(x)$  satisfy both equations  $\mathcal{E}_\mu$  and  $\mathcal{E}_\lambda$ . Notice that since the l.h.s.  $\tilde{\Lambda}_\delta(x)$  of (4.6) vanishes at  $x = 0$ , there exists a branch  $h(x)$  which vanishes at  $x = 0$ . Moreover, its behaviour is  $h(x) = \tilde{\Lambda}_\delta(x)/C_2 + O(\tilde{\Lambda}_\delta^2(x))$ . Further,  $\tilde{\Lambda}_\delta(x) = 2(B_+ - A_+)x + O(x^3)$  in the case  $B_+ \neq A_+$ , and  $\tilde{\Lambda}_\delta(x) = \frac{8}{3}B_+x^3 + O(x^5)$  otherwise. In any case, this branch  $h(x)$  is regular at  $x = 0$ .

Now consider the equations  $\mathcal{E}_\mu$ . By the consideration above, it has the form  $h_x(h_x^2 - h(x)^2 + A_2) = 2A_+\cosh(x)$ . This equation has (at most) three local branches of solutions satisfying the initial value problem  $h|_{x=0} = 0$  each corresponding to a root of the equation  $\mathcal{E}_\mu|_{x=0}$  considered as a cubic polynomial on  $h_x|_{x=0}$ . However, we can immediately see that these solutions are  $2a^{(i)}\sinh(x)$  where  $a^{(i)}$  are three roots of the polynomial equation  $a \cdot (4a^2 + A_2) = A_+$ .<sup>2</sup> The uniqueness of solutions of the initial value problem for ODEs implies the equality  $h(x) = 2a^{(i)}\sinh(x)$ . However, this contradicts to the non-constancy of  $R = h_{xxx}h_x - h_{xx}^2$ . So finally we have excluded the possibility  $\lambda = \pm 3\mu$  with  $B_+ \neq 0 \neq B_-$ .

Let us notice that transforming the above solution  $h(x) = 2a\sinh(x)$  by means of affine change of the coordinate  $x$  ( $x \mapsto b(x + x_0)$  with  $b \neq 0, x_0 \in \mathbb{C}$ ) and the function itself ( $h \mapsto a'h + c$  with  $a' \neq 0, c \in \mathbb{C}$ ) we can obtain all solutions given in items (1–3) of Theorem 3.1.

The next case we consider is when  $B_- = 0$ . Recall that we also have  $\mu = 1$  and  $\lambda = 3$ . We shall consider the asymptotic behaviour of various expressions for  $x$  varying some on a real line in  $\mathbb{C}$  given by  $\Im(x) = c$  and tending to  $i \cdot c - \infty$ . To simplify notation, we write this as  $x \rightarrow -\infty$ . Assume that  $A_- \neq 0$ . Then from (4.6) we obtain the asymptotic growth  $h(x) \sim -\left(\frac{3A_-}{C_0}\right)^{1/3}e^{-x/3}$  for  $x \rightarrow -\infty$  and from (4.5) a similar growth of the derivative  $h_x$ . The substitution of this asymptotic in the l.h.s. of the equation  $\mathcal{E}_\mu$  would give the growth  $\sim C \cdot e^{-x}$  for  $x \rightarrow -\infty$ , whereas the l.h.s.  $\mathcal{E}_\mu$  decreases. The contradiction shows that we must have  $A_- = 0$ . Now from (4.6) we conclude that for  $x \rightarrow -\infty$  the function  $h(x)$  is given by the converging series  $h(x) = \sum_{j=0}^{\infty} a_j e^{jx}$  with some complex coefficients  $a_j$ . The substitution of this series in the equations gives  $a_1(B_2 - 9a_0^2)e^x + O(e^{2x}) = 0$  for  $\mathcal{E}_\lambda$ , and  $(A_2a_1 - a_0^2a_1 - A_+)e^x + O(e^{2x}) = 0$  for  $\mathcal{E}_\mu$ . In the case  $a_1 = 0$  we would have  $A_+ = 0$  and hence  $\Lambda_\lambda \equiv 0$ , which was excluded above. Consequently,  $B_2 = 9a_0^2$ . Substituting this relation in  $\mathcal{E}_\lambda$ , we consider the further expansion of  $\mathcal{E}_\lambda$ . This gives us  $a_1a_0^2e^{2x} + O(e^{3x}) = 0$ , and hence  $a_0 = 0$  since by the above argument  $a_1 \neq 0$ . Repeating the substitution we obtain  $-(B_+ + 8a_1^3)e^{3x} + O(e^{4x}) = 0$  from  $\mathcal{E}_\lambda$ , and  $(A_2a_1 - A_+)e^x + O(e^{2x}) = 0$  for  $\mathcal{E}_\mu$ . This gives us  $B_+ = -8a_1^3$  and  $A_+ = a_1A_2$ . Now we see that for each of three roots  $a_1$  of the equation  $8a_1^3 + B_+ = 0$  the function  $a_1e^x$  satisfies the ODE  $\mathcal{E}_\mu$  and has the correct asymptotic behaviour for  $x \rightarrow -\infty$ . Consequently,  $h(x)$  is one of these three solutions, and hence  $R = h_{xxx}h_x - h_{xx}^2$  must be constant. The obtained contradiction excludes the possibility  $B_- = 0$ .

<sup>2</sup> The case when this polynomial has a multiple root is degenerate: In this case the discriminant of  $\mathcal{E}_\mu$  with respect to  $h_x$  vanishes identically along the corresponding solution  $h(x)$ . Moreover, solving the corresponding initial value problem in the form of series  $h(x) = \sum_j c_j x^j$  with  $c_0 = 0$  we obtain the cubic equation on  $c_1 \cdot (4c_1^2 + A_2) = A_+$  which has one double and one simple root. Substituting this double root in  $c_1$ , all successive equations on  $c_2, c_3, \dots$  can be solved uniquely. This gives us the uniqueness of the problem also for this degenerate case.

Above we have proven the equality  $\lambda = \mu$  under hypotheses of the theorem and additional assumption  $A_0 \neq 0 \neq B_0$ . Now we consider the case when one of these coefficients vanishes, say  $B_0 = 0$ . Then  $B_1 \neq 0$  since otherwise we obtain the equation  $B_0 h_x = \Lambda_\lambda$  whose solutions are  $h(x) = c_+ e^{\mu x} + c_- e^{-\mu x} + c_0$  (or  $h(x) = c_2 x^2 + c_1 x + c_0$  in the case  $\mu = 0$ ) for which  $R = h_{xxx} h_x - h_{xx}^2$  would be constant. Making transformations  $h \mapsto ah + c$  and  $x \mapsto x/\lambda$  in the case  $\lambda \neq 0$  we change the equation  $\mathcal{E}_\lambda$  into  $h(x)h_x = B_+ e^{2x} + B_- e^{-2x}$  (which means that we make  $\lambda = 2$ ) or respectively  $h(x)h_x = B_3 x + B_4$ . The integration gives  $h(x)^2 = B_+ e^{2x} - B_- e^{-2x} + B_5$  or resp.  $h(x)^2 = B_3 x^2 + 2B_4 x + B_5$  with some  $B_5 \in \mathbb{C}$ . As it was shown above without loss of generality we can suppose that  $B_+ \neq 0$ . Hence we conclude that in the case  $\lambda = 2$  the function  $h(x)$  is given by a series  $h(x) = ae^x + \sum_{j=0}^{\infty} a_j e^{-(2j+1)x}$  which converges for  $\Re(x) \geq x_0 \gg 0$  and such that  $a^2 = B_+$ . In the case  $\lambda = 0$  we obtain respectively  $h(x) = \pm \sqrt{B_3 x^2 + 2B_4 x + B_5} \sim \pm \sqrt{B_3} x$  in the case  $\lambda = 0$ . In particular, the asymptotic for the derivative is  $h_x \sim B_+^{1/2} e^x$  or respectively  $h_x \sim \pm \sqrt{B_3}$ . The substitution in  $\mathcal{E}_\mu$  and comparing of the growth of the left and right hand sides exclude the case  $\lambda = 0$  and shows that we could have  $\mu = \pm 1$ ,  $\mu = \pm 2$ , or  $\mu = \pm 3$  in the case  $\lambda = 2$ . The case  $\mu = \pm 2 = \pm \lambda$  is our claim, so we must exclude two other possibilities.

First, we consider the case  $\lambda = 2$  and  $\mu = \pm 1$ . As above we distinguish the subcases  $B_- \neq 0$  and  $B_- = 0$  and start with the first one  $B_- \neq 0$ . Shifting the coordinate  $x$  appropriately we make  $B_- = -B_+$ . Then the coefficients  $B_+, B_5$  and  $a, a_0, a_1, \dots$  are related as

$$B_+ = a^2 \quad B_5 = -2aa_0 \quad a_1 = \frac{a^2 - a_0^2}{2a} \quad a_2 = -\frac{a_0(a^2 - a_0^2)}{2a^2} \quad a_3 = -\frac{(a^2 - a_0^2)(a^2 - 5a_0^2)}{8a^3}, \quad (4.8)$$

and so on. Substitute the series  $h(x) = ae^x + \sum_{j=0}^{\infty} a_j e^{-(2j+1)x}$  and the above relation in  $\mathcal{E}_\mu$  and write the condition of the vanishing of the resulting expansion. We obtain subsequently  $A_+ = aA_2 - 4a^2 a_0$ ,  $A_- = 4aa_0^2 - 8a^2 a_1 - A_2 a_0 = 8aa_0^2 - A_2 a_0 - 4a^3$ , and then the condition  $\frac{3(a^2 - a_0^2)(A_2 - 12aa_0)}{2a} = 0$ . So here we have two possibilities: either  $A_2 = 12aa_0$  or  $a_0 = \pm a$  (or both). However, after substitution of the first relation  $A_2 = 12aa_0$  in  $\mathcal{E}_\mu$  the first non-trivial term will be  $\frac{10(a^2 - a_0^2)^2}{a} e^{-5x}$  which leads to the relation  $a_0 = \pm a$  dropped above. But then all higher coefficients  $a_1, a_2, \dots$  must vanish and the solution  $h(x)$  of the equation  $h(x)^2 = B_+ e^{2x} - B_- e^{-2x} + B_5$  will be  $2a \cosh(2x)$  in the case  $a_0 = +a$  or respectively  $2a \sinh(2x)$  in the case  $a_0 = -a$ . In any case  $R = h_{xxx} h_x - h_{xx}^2$  will be constant. The obtained contradiction excludes the possibility  $\lambda = \pm 2\mu$ ,  $B_0 = 0$ , and  $B_- \neq 0$ .

Our next subcase is  $\lambda = 2$ ,  $\mu = 1$ , and  $B_0 = B_- = 0$ . The procedure here is essentially the same as in the previous subcase: Substituting the series  $h(x) = ae^x + \sum_{j=0}^{\infty} a_j e^{-(2j+1)x}$  in  $h(x)^2 = B_+ e^{2x} + B_5$  we obtain the relations

$$B_+ = a^2 \quad B_5 = -2aa_0 \quad a_1 = -\frac{a_0^2}{2a} \quad a_2 = \frac{a_0^3}{2a^2} \quad a_3 = -\frac{5a_0^4}{8a^3}, \quad (4.9)$$

and so on. Next we substitute the series  $h(x) = ae^x + \sum_{j=0}^{\infty} a_j e^{-(2j+1)x}$  and the obtained relations in  $\mathcal{E}_\mu$  and get  $A_+ = aA_2 - 4a^2 a_0$ ,  $A_- = 8aa_0^2 - A_2 a_0$ , and then the condition  $\frac{3a_0^2(A_2 - 12aa_0)}{2a} = 0$ . As before, setting  $A_2 = 12aa_0$  in  $\mathcal{E}_\mu$  we then obtain  $\frac{10a_0^4}{a} = 0$  which gives us the condition  $a_0 = 0$  dropped before. So we must have  $a_0 = 0$  and  $h(x) = ae^x$ , and hence  $R = h_{xxx} h_x - h_{xx}^2$  will vanish identically. The contradiction excludes also this subcase.

Next we consider the case  $\lambda = 2$  and  $\mu = \pm 3$  and start with the subcase one  $B_- \neq 0$ . As in the case  $\mu = \pm 1$  above we can additionally assume  $B_- = -B_+$ . Then we obtain the same expansion  $h(x) = ae^x + \sum_{j=0}^{\infty} a_j e^{-(2j+1)x}$  with the same relations (4.8). Substituting

them in  $\mathcal{E}_\mu$  we obtain subsequently the relations  $A_+ = -8a^3$ ,  $A_1 = 0$ ,  $A_2 = 12aa_0$ ,  $A_- = 12a^2a_0 - 4a_0^3$  and then the condition  $\frac{15(a^2 - a_0^2)^2}{a} = 0$ . As above, in both cases  $a_0 = \pm a$  all higher coefficients  $a_1, a_2, a_3, \dots$  vanish, the solution  $h(x)$  must be either  $2a\cosh(x)$  or  $2a\sinh(x)$ , and the function  $R = h_{xxx}h_x - h_{xx}^2$  will be constant.

In the subcase  $\lambda = 2$  and  $\mu = \pm 3$  and  $B_- = 0$  we obtain respectively first the relations (4.9), then subsequently the relations  $A_+ = -8a^3$ ,  $A_1 = 0$ ,  $A_2 = 12aa_0$ ,  $A_- = -4a_0^3$ , and then the condition  $\frac{15a_0^4}{a} = 0$ . The rest follows as in the case  $\lambda = 2$ ,  $\mu = 1$ , and  $B_0 = B_- = 0$  considered above.

This finishes the proof of the fact that under the hypotheses of the theorem one has the relation  $\mu = \pm\lambda$ . Now we show the complete assertion, namely, the uniqueness of the equation up to constant factor. As before, we suppose that  $h(x)$  satisfies two equations, for which we maintain the above notation  $\mathcal{E}_\mu, \mathcal{E}_\lambda, \Lambda_\mu, \Lambda_\lambda, A_0, \dots, B_0, \dots, A_\pm, B_\pm$ . Besides, we may assume the equality  $\mu = \lambda$ .

Since  $\mu = \lambda$ , a linear combination of  $\mathcal{E}_\mu$  and  $\mathcal{E}_\lambda$  is again an equation of the same form with the same  $\mu$ . In particular, we can replace  $\mathcal{E}_\mu$  or  $\mathcal{E}_\lambda$  by such a linear combination. Consequently, we can assume that  $B_0 = 0$ , and in the case  $A_0 = B_0 = 0$  we may also suppose that  $B_1 = 0$ . However, in the latter case we would have  $B_2h_x = B_+e^{\mu x} + B_-e^{-\mu x}$  (resp.  $B_2h_x = B_3x + B_4$  in the case  $\mu = 0$ ) and hence  $R = h_{xxx}h_x - h_{xx}^2$  would be constant. The contradiction shows that we must have  $A_0 \neq 0 \neq B_1$ . Normalising, we can make  $A_0 = 1 = B_1$ .

First, let us consider the case  $\mu = \lambda \neq 0$ . Here we apply essentially the same arguments as in the above cases  $\lambda = 2, \mu = 1$  and  $\lambda = 2, \mu = 3$ . As we have shown above, making appropriate transformations the equation  $\mathcal{E}_\lambda$  can be brought to the form  $h(x)h_x = B_+e^{2x} + B_-e^{-2x}$  with  $B_+ \neq 0$ , in particular, we make  $\mu = \lambda = 2$ . In this way we obtain the algebraic equation  $h(x)^2 = B_+e^{2x} - B_-e^{-2x} + B_5$  and the asymptotic growth  $h(x) = ae^x + O(e^{-x})$  and  $h_x = ae^x + O(e^{-x})$  for  $x \rightarrow +\infty$  with  $a^2 = B_+ \neq 0$ . The substitution gives the growth  $-3a^3e^{3x} + O(e^{2x})$  of the l.h.s. of  $\mathcal{E}_\mu$ , which contradicts to  $\Lambda_\mu = A_+e^{2x} + A_-e^{-2x}$ .

The argumentation in the case  $\mu = \lambda = 0$  is as follows. The equation  $h(x)h_x = B_3x + B_4$  integrates to  $h(x)^2 = B_3x^2 + 2B_4x + B_5$ . An appropriate affine transformation of  $x$  and a rescaling of  $h$  bring this equation into one of the following forms:  $h(x)^2 = x^2 + 1$ ,  $h(x)^2 = x^2$ ,  $h(x)^2 = x$ , or  $h(x)^2 = 1$ . In the cases  $h(x)^2 = x^2$  and  $h(x)^2 = 1$  the expression  $R = h_{xxx}h_x - h_{xx}^2$  vanishes in contradiction to the hypothesis of the theorem. In the remaining cases the function  $h(x)$  can not satisfy the equation  $h_x(h_x^2 - A_1h(x) + A_2) = A_3x + A_4$ .

It remains to consider the case  $A_0 = 0 = B_0$  ("Darboux-superintegrable case"). Then both  $A_1$  and  $B_1$  must be non-zero since otherwise  $R = h_{xxx}h_x - h_{xx}^2$  would be constant as we have shown above. Normalisation of the equations transforms them into  $h_x(h(x) + A_2) = A_+e^{\mu x} + A_-e^{-\mu x}$  (or  $= A_3x + A_4$  in the case  $\mu = 0$ ) and respectively  $h_x(h(x) + B_2) = B_+e^{\lambda x} + B_-e^{-\lambda x}$ . The subsequent integration gives

$$\begin{aligned} \frac{h(x)^2}{2} + A_2h(x) &= \frac{A_+}{\mu}e^{\mu x} - \frac{A_-}{\mu}e^{-\mu x} + A_5, \\ \frac{h(x)^2}{2} + A_2h(x) &= \frac{A_3}{2}x^2 + A_4x + A_5 \quad \text{in the case } \mu = 0, \\ \frac{h(x)^2}{2} + B_2h(x) &= \frac{B_+}{\lambda}e^{\lambda x} - \frac{B_-}{\lambda}e^{-\lambda x} + B_5. \end{aligned} \tag{4.10}$$

In the case  $A_+ = A_- = 0$  the function  $h(x)$  must be constant which contradict the hypotheses of the theorem. So one of these coefficient must be non-zero, and changing the sign of  $\mu$  if

needed we can suppose that  $A_+ \neq 0$ . By the same argument  $B_+$  is non-zero. Observe that the equations 4.10 establish an algebraic dependence between the functions  $e^{\mu x}$  and  $e^{\lambda x}$  in the case  $\mu \neq 0 \neq \lambda$ , and between the functions  $x$  and  $e^{\lambda x}$  in the case  $\mu = 0 \neq \lambda$ . This can be possible only if  $\mu = \pm\lambda$ . In this situation the difference of the integrated equations 4.10 is

$$(A_2 - B_2)h(x) = \frac{A_+ - B_+}{\mu} e^{\mu x} - \frac{A_- - B_-}{\mu} e^{-\mu x} + (A_5 - B_5)$$

or respectively

$$(A_2 - B_2)h(x) = \frac{A_3 - B_3}{2} x^2 + (A_4 - B_4)x + (A_5 - B_5)$$

in the case  $\lambda = \mu = 0$ . Now it is obvious that the triviality of these relations is the only possibility to avoid the contradiction with the condition  $R = h_{xxx}h_x - h_{xx}^2 \neq \text{const}$ . This means the desired proportionality of the equations.

The theorem is proved. □

**4.1. Real solutions.** Recall that the Principal equations (2.8), (2.13) have the following meaning: If a surface metric  $g$  admits a linear and a non-trivial cubic integral then it has the form  $h_x^{-2}(dx^2 + dy^2)$  with a function  $h(x)$  satisfying one of these two equations with some *complex* parameters  $\mu, A_0, \dots, A_4$ . Of course, we are interested only in solutions for which  $h_x$  is real. In this case  $h(x) = h_1(x) + i \cdot c$  with some *real* function  $h_1(x)$  and a real constant  $c$ . Substituting we see that  $h_1(x)$  satisfies the same equation with new parameters  $A_0, \dots, A_4$ . Thus we can consider only real solutions  $h(x)$ .

**Theorem 4.2.** *Assume that the equation (1.2) with some complex parameter  $\mu$  and complex coefficients admits a real-valued solution  $h(x)$  such that  $R = h_{xxx} \cdot h_x - h_{xx}^2$  is non-constant. Then  $\mu$  is real or purely imaginary (or zero) and the equation is complex proportional to another equation (1.2) with the same parameter  $\mu$  and with real coefficients  $A_0, \dots, A_4$ .*

*Proof.* The result follows immediately from Theorem 4.1 applied to the equation (1.2) and its complex conjugate. □

## 5. NUMBER OF INDEPENDENT CUBIC INTEGRALS.

### PROOF OF KRUGLIKOV'S "BIG GAP" CONJECTURE.

### SUMMARY OF THE PROOF OF THE MAIN THEOREM.

**5.1. Number of cubic integrals and Kruglikov's "big gap" conjecture.** In [Kr] Kruglikov conjectured that the dimension of the space of cubic integrals of a surface metric  $g$  of non-constant curvature is at most 4. In this section we prove this result for metrics satisfying the hypotheses of the main theorem. Our proof applies also for Darboux-superintegrable metrics, however, the result in the case is not new.

**Theorem 5.1.** *Let a function  $h(x)$  satisfy one of the equations (1.2) with complex parameters  $\mu, A_0, \dots, A_4$ . Assume that  $R := h_{xxx}h_x - h_{xx}^2$  is non-constant. Set  $H := \frac{1}{2}h_x^2(p_x^2 + p_y^2)$ . Then the space of complex-valued functions  $F(x, y; p_x, p_y)$  that are cubic in momenta  $(p_x, p_y)$  and satisfy the equation  $\{H, F\} = 0$  is 4-dimensional (as vector space over  $\mathbb{C}$ ).*

*Proof.* We distinguish two main cases:  $\mu \neq 0$  and  $\mu = 0$  and start with the first one. Set  $L := p_y$ . Then  $\{H, L\} = 0$ . This gives us the following 4 linearly independent solutions of the equation  $\{H, F\} = 0$ :  $L^3$ ,  $H \cdot L$ , and 2-dimensional space of solutions  $F$  given by the formulas (1.3).<sup>3</sup> So the theorem claims that there are no more linearly independent solutions.

We call functions  $F(x, y; p_x, p_y)$  satisfying the hypotheses of the theorem (complex) cubic integrals (of the Hamiltonian  $H$  given by the function  $h$ ). Denote by  $\mathcal{F}_h$  the space of complex cubic integrals. It was shown by Kruglikov [Kr] that the space  $\mathcal{F}_h$  is finite-dimensional.<sup>4</sup> The Jacobi identity implies that the formula  $\mathcal{L} : F \mapsto \{L, F\}$  induces a well defined homomorphism  $\mathcal{L} : \mathcal{F}_h \rightarrow \mathcal{F}_h$ , see §2.1. Consider the decomposition of  $\mathcal{F}_h$  into generalised eigenspaces of  $\mathcal{L}$  and the corresponding Jordan blocks. Then  $L^3$  and  $H \cdot L$  are eigenvectors with eigenvalue 0. Further, the functions  $F_+$  and respectively  $F_-$  given by formula (1.3) with  $C_- = 0$  and respectively  $C_+ = 0$  are eigenvectors of  $\mathcal{L}$  with eigenvalues  $\pm\mu$ .

It follows immediately from Theorem 4.1 that the space  $\mathcal{F}_h$  contains no eigenvectors of  $\mathcal{L}$  with eigenvalue  $\lambda \neq \pm\mu$ . Thus in the case  $\mu \neq 0$  the assertion of the theorem is equivalent to the non-existence of a generalised eigenvector of  $\mathcal{L}$  with eigenvalue  $\pm\mu$  and the Jordan block  $\begin{pmatrix} \pm\mu & 1 \\ 0 & \pm\mu \end{pmatrix}$ .

Assume the contrary. Then there would exist cubic integrals  $F_0, F_1 \in \mathcal{F}_h$  satisfying  $\{L, F_1\} = \pm\mu F_1 + F_0$  and  $\{L, F_0\} = \pm\mu F_0$ . Inverting the  $y$ -axis we can change the sign. So we assume that we have  $+\mu$  in the formulas. Recall that  $L = p_y$  correspond to the vector field  $\frac{\partial}{\partial y}$ . Integrating the equations above we obtain  $F_0 = e^{\mu y} G_0$  and  $F_1 = y e^{\mu y} G_0 + e^{\mu y} G_1$  where  $G_0, G_1$  are some complex functions of  $(x, p_x, p_y)$  cubic in momenta  $(p_x, p_y)$  and independent of  $y$ . Since  $F_0$  is a cubic integral and an eigenvector of  $\mathcal{L}$  with eigenvalue  $\mu$ , it has the form (1.3). Since  $\{F_0, H\} = \{H, e^{\mu y} G_0\} = 0$ , the equation  $\{F_1, H\} = 0$  now reads  $\{e^{\mu y} G_1, H\} + \{y, H\} e^{\mu y} G_0 = 0$ . Write  $G_1 = \sum_{j=0}^3 b_j(x) p_x^{3-j} p_y^j$ . Since  $\{y, H\} = -p_y h_x^2$ , we obtain the equation

$$\{e^{\mu y} G_1, H\} - p_y h_x^2 e^{\mu y} G_0 = 0 \quad (5.1)$$

Solving this equation we apply the same procedure as in §2.2. The bracket  $\{e^{\mu y} G_1, H\}$  is given by (2.4) in which we need to replace  $a_j(x)$  by  $b_j(x)$ . Thus the equation (5.1) is equivalent to 5 equations which are *inhomogeneous* versions of 5 equations in (2.4) with the r.h.s.-s given by  $p_y h_x^2 e^{\mu y} G_0$ . As in §2.2 we solve successively the first 3 of them and resolve  $b_3(x)$  from the last one. This gives the following formulas (compare with (2.5)):

$$\begin{aligned} b_0(x) &= B_0 h_x^3 \\ b_1(x) &= (-\mu B_0 + A_0) \cdot h(x) + \frac{B_1}{2\mu} \cdot h_x^2 \\ b_2(x) &= \frac{1}{2} \cdot \left( -(B_1 + \frac{A_1}{\mu}) \cdot h(x) + (\mu^2 B_0 + 2\mu A_0) \cdot h(x)^2 + 3B_0 h_x^2 + B_2 \right) \cdot h_x \\ b_3(x) &= \frac{1}{2\mu^2} \cdot \left( 3 \cdot h_x^2 \cdot (\mu B_0 - A_0) - B_1 \cdot h(x) + \mu^2 \cdot (\mu B_0 + A_0) \cdot h(x)^2 + (\mu B_2 - A_2) \right) \cdot h_{xx}, \end{aligned} \quad (5.2)$$

Substituting them in the remaining term of (5.1) we obtain the equation

$$\begin{aligned} & (3 \cdot (\mu B_0 - A_0) \cdot h_x^2 + \mu^2 \cdot (\mu B_0 + A_0) \cdot h(x)^2 - \mu \cdot B_1 \cdot h(x) + \mu \cdot B_2 - A_2) \cdot h_{xxx} \\ & + 6 \cdot (\mu B_0 - A_0) \cdot h_x \cdot h_{xx}^2 + (6 \cdot \mu^2 \cdot (A_0 + \mu \cdot B_0) \cdot h(x) - 3 \cdot \mu \cdot B_1) \cdot h_x \cdot h_{xx} \\ & + 63 \cdot \mu^2 \cdot (A_0 + \mu \cdot B_0) \cdot h_x^3 \\ & + (\mu^4 \cdot (\mu \cdot B_0 + 3 \cdot A_0) \cdot h(x)^2 - \mu^2 \cdot (\mu \cdot B_1 + 2 \cdot A_1) \cdot h(x) + \mu^2 \cdot (\mu \cdot B_2 + A_2)) \cdot h_x = 0 \end{aligned} \quad (5.3)$$

<sup>3</sup>The fact that the parameters  $\mu, A_0, \dots, A_4$  in Theorem 1.1 are real plays no role here.

<sup>4</sup>The proof in [Kr] is given for the case  $H(x, y; p_x, p_y) = (dx^2 + dy^2)/\lambda(x, y)$  with *real*  $\lambda(x, y)$ . It works in our situation without changes.

which is the counterpart of (2.6). As the equation (2.6), the above equation can be partially integrated in the sense that it can be written in the form (compare with (2.7))

$$\begin{aligned} & \mu \cdot \left( \frac{d^2}{dx^2} + \mu^2 \right) \left( h_x \cdot (h_x^2 \cdot B_0 + (\mu^2 \cdot B_0 + 2\mu \cdot A_0) \cdot h(x)^2 - (B_1 + \frac{A_1}{\mu}) \cdot h(x) + B_2) \right) \\ & + \left( \frac{d^2}{dx^2} - \mu^2 \right) \left( h_x \cdot (h_x^2 \cdot A_0 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) \right) = 0. \end{aligned} \quad (5.4)$$

Let us now observe that the other equation  $\{F_0, H\} = \{H, e^{\mu y} G_0\} = 0$  is equivalent to the equation  $(h_x \cdot (h_x^2 \cdot A_0 + \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2)) = A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x)$  with some constants  $A_3, A_4$ , and that the l.h.s. of this equation appears in (5.4). Then

$$\left( \frac{d^2}{dx^2} - \mu^2 \right) (A'_3 \sin(\mu x) + A_4 \cos(\mu x)) = -2\mu^2 (A'_3 \sin(\mu x) + A_4 \cos(\mu x))$$

(here we set  $A'_3 = \frac{A_3}{\mu}$ ) and so (5.4) is equivalent to

$$\begin{aligned} & h_x \cdot (h_x^2 \cdot B_0 + (\mu^2 \cdot B_0 + 2\mu \cdot A_0) \cdot h(x)^2 - (B_1 + \frac{A_1}{\mu}) \cdot h(x) + B_2) = \\ & (B_3 + A_4 \cdot x) \cdot \sin(\mu x) - (A'_3 \cdot x + B_4) \cdot \cos(\mu x) \end{aligned} \quad (5.5)$$

For convenience in future let us make the substitution  $\mu \mapsto i\mu$ ,  $B_i \mapsto iB_i$  in the equations (5.5) and (2.8), and rearrange their r.h.s.-s. Then the equations transform into

$$h_x \cdot (h_x^2 \cdot A_0 - \mu^2 \cdot A_0 \cdot h(x)^2 - A_1 \cdot h(x) + A_2) = A_+ \cdot e^{\mu x} + A_- \cdot e^{-\mu x} \quad (5.6)$$

$$\begin{aligned} & h_x \cdot (h_x^2 \cdot B_0 - (\mu^2 \cdot B_0 - 2\mu \cdot A_0) \cdot h(x)^2 - (B_1 - \frac{A_1}{\mu}) \cdot h(x) + B_2) = \\ & (B_+ + A_+ \cdot x) \cdot e^{\mu x} + (B_- - A_- \cdot x) \cdot e^{-\mu x} \end{aligned} \quad (5.7)$$

Notice that the condition of non-triviality of  $F_0$  is equivalent to the non-vanishing of at least one parameter  $A_0, A_1, A_2$ . Further, by Lemma 4.2 both  $A_+$  and  $A_-$  can not vanish together. Replacing  $\mu$  by  $-\mu$ , if needed, we can suppose that  $A_+ \neq 0$ .

We consider several subcases. The first one is  $A_0 = 0$ . Then the equation (5.6) can be integrated as

$$-\frac{A_1}{2} h(x)^2 + A_2 h(x) = \frac{A_+}{\mu} \cdot e^{\mu x} - \frac{A_-}{\mu} \cdot e^{-\mu x} + A_5 \quad (5.8)$$

If, moreover,  $A_1 = 0$ , then  $h(x) = \frac{A_+}{\mu A_2} \cdot e^{\mu x} - \frac{A_-}{\mu A_2} \cdot e^{-\mu x} + \frac{A_5}{A_2}$ , and then  $R = \text{const}$  in this case in contradiction with the hypotheses of the theorem. Otherwise we make the substitution  $x \mapsto 2x/\mu$ . After the substitution  $\mu$  transforms into 2 and the r.h.s. of (5.8) into  $\frac{A_+}{2} \cdot e^{2x} - \frac{A_-}{2} \cdot e^{-2x} + A_5$ . So we can conclude that for  $x \rightarrow +\infty$  the function  $h(x)$  is given by the converging series  $ae^x + \sum_{j=0}^{\infty} a_j e^{-jx}$ . But then the substitution of this series in 5.7 gives the following leading terms for  $x \rightarrow +\infty$ :  $-3a^3 e^{3x}$  for the l.h.s., and  $\frac{2A_+}{\mu} x e^{2\mu}$  for the r.h.s. The obtained contradiction shows that the case  $A_0 = 0$  is impossible.

In the case  $A_0 \neq 0$  we make the substitution  $x \mapsto 3x/\mu$  which makes  $\mu = 3$ , and subtract (5.6) from (5.7) with coefficients  $B_0/A_0$ . This gives us the equation

$$h_x (3C_0 h(x)^2 + 2C_1 h(x) + C_2) = A_+ x e^{3x} + C_4 e^{3x} + (C_5 x + C_6) e^{-3x} \quad (5.9)$$

with some constants  $C_0, C_1, \dots$  such that  $C_0 \neq 0$ . Integrating it, we obtain

$$C_0 h(x)^3 + C_1 h(x)^2 + C_2 h(x) + C_3 = \frac{A_+}{3} x e^{3x} + \tilde{C}_4 e^{3x} + (\tilde{C}_5 x + \tilde{C}_6) e^{-3x} \quad (5.10)$$

with some new constants  $C_3, \tilde{C}_5, \tilde{C}_6$ . From this equation we conclude that  $h(x)$  is a 3-sheeted function on  $\mathbb{C}$  with a ramification on some discrete subset  $S \subset \mathbb{C}$  and that for  $x \rightarrow +\infty$  every branch of  $h(x)$  has a behaviour  $h(x) = ax^{1/3} e^x (1 + \Lambda(x))$  for some function  $\Lambda(x)$  admitting a converging series  $\sum_{ij=0}^{\infty} a_{ij} e^{-ix} x^{-j/3}$  with  $a_{00} = 0$ . Moreover, the derivative  $h_x$  is given by the derivative of the expansion above and is a similar series  $h_x = ax^{1/3} e^x (1 + \tilde{\Lambda}(x))$  with

$\tilde{\Lambda}(x) = \sum_{ij=0}^{\infty} \tilde{a}_{ij} e^{-ix} x^{-j/3}$  such that  $\tilde{a}_{00} = 0$ . Substituting these expansions in (5.6) we obtain the term  $-8a^3 x e^{3x}$  for the l.h.s., which contradicts to the growth  $A_+ e^{3x}$  of r.h.s.

This prohibits the possibility  $A_0 \neq 0$  for a solution  $h(x)$  of the pair of equations (5.6)–(5.7), and thus excludes cubic integrals  $F_1, F_0$  such that  $\{L, F_1\} = F_0$  and  $\{L, F_0\} = 0$ .

Now we consider the case  $\mu = 0$ . As above, denote  $L = p_x$ , set  $\mathcal{L}(F) := \{L, F\}$  for any function  $F(x, y; p_x, p_y)$  and let  $\mathcal{F}_h$  be the space of complex cubic integrals oh  $H$ . Then as above  $\mathcal{F}_h$  is finite dimensional and  $\mathcal{L} : \mathcal{F}_h \rightarrow \mathcal{F}_h$  is a well-defined homomorphism. It follows from Theorem 4.1 that in the case  $\mu = 0$  the homomorphism  $\mathcal{L} : \mathcal{F}_h \rightarrow \mathcal{F}_h$  has unique eigenvalue  $\mu = 0$  and  $\mathcal{F}_h$  is a sum of Jordan blocks with eigenvalue  $\mu = 0$ .

Notice that  $L^3$  and  $L \cdot H$  are eigenvectors of  $\mathcal{L}$  with eigenvalue  $\mu = 0$ . We are going to prove that  $\mathcal{F}_h$  contains only two linearly independent eigenvectors, a unique Jordan block of size  $3 \times 3$ , and no other Jordan blocks.

Let  $F_0 = F_0(x, y; p_x, p_y)$  be given by (2.2) with coefficients  $a_0(x), \dots, a_3(x)$  given by (2.10) with  $\tilde{A}_1 = \tilde{A}_3 = 0$ . Then  $F_0$  is a cubic integral,  $\mathcal{L}(F_0) = A_1 \cdot L^3 + A_3 \cdot L \cdot H \neq 0$ , and  $\mathcal{L}^2(F_0) = 0$ . Assume that we have some other non-zero cubic integral  $F'$  such that  $\mathcal{L}^2(F') = 0$ . Then the calculation made in § 2.1 and § 2.3 show that  $F_0$  must be given by the same (2.2) with new coefficients  $a'_0(x), \dots, a'_3(x)$  given by (2.10) with parameters  $A'_0, \dots, A'_3$  instead of  $A_0, \dots, A_3$ , such that  $h(x)$  satisfies the Principle equation (1.2) (iii) with  $A_i$  replaced by  $A'_i$ .

At this point we obtain two subcases. The first is when both  $A'_1, A'_3$  vanish, which means that  $\mathcal{L}(F') = 0$ , i.e.,  $F'$  is an eigenvector of  $\mathcal{L}$  with eigenvalue  $\mu = 0$ . In this situation from the equation (1.2) (iii) we see that either  $h_x$  is constant or all parameters  $A'_0, \dots, A'_4$  must vanish. The first possibility would yield  $R = h_{xxx} h_x - h_{xx}^2 \equiv 0$ , which contradicts the hypotheses of the theorem. Thus all parameters  $A'_0, \dots, A'_4$  must vanish, and then  $F' = \tilde{A}'_1 \cdot L^3 + \tilde{A}'_3 \cdot L \cdot H$ , a linear combination of  $L^3$  and  $L \cdot H$ .

Let us underline, that the latter argument demonstrates that the space of eigenvectors of  $\mathcal{L}$  in  $\mathcal{F}_h$  is 2-dimensional with a basis  $L^3, L \cdot H$ .

The remaining subcase is when not all parameters  $A'_0, \dots, A'_4$  vanish and we obtain a new equation of the form (1.2) (iii). In this situation the uniqueness from Theorem 4.1 ensures that  $A'_i = c \cdot A_i$  with some coefficient  $c$ . But in this case  $F' = c \cdot F_0 + \tilde{A}'_1 \cdot L^3 + \tilde{A}'_3 \cdot L \cdot H$  with the same coefficient  $c$  and some parameters  $\tilde{A}'_1, \tilde{A}'_3$ . This demonstrates that the space of cubic integrals  $F'$  satisfying  $\mathcal{L}^2(F') = 0$  is 3-dimensional with a basis  $F_0, L^3, L \cdot H$ . In particular, we can not have two distinct Jordan blocks.

Finally, let us show that there does exist a Jordan block of size  $3 \times 3$ , and no Jordan block of size  $4 \times 4$ . For this purpose we try to find a cubic integral  $F$  satisfying  $\mathcal{L}^4(F) = 0$ . Since the operator  $\mathcal{L}$  acts as the derivation in  $y$ , the condition  $\mathcal{L}^4(F) = 0$  means that  $F$  is a polynomial in  $y$  of degree  $\leq 3$ . This means that we can write  $F$  in the form

$$F = \sum_{i=0}^2 \sum_{j=0}^3 a_{ij}(x) y^i p_x^{3-j} p_y^j \quad (5.11)$$

with some coefficients  $a_{ij}(x)$ . Writing down the equation  $\{F, H\} = 0$  and considering its coefficients at monomials  $y^i p_x^{4-j} p_y^j$  we obtain 15 ODEs on functions  $a_{ij}(x)$  and  $h(x)$ . We solve them subsequently using the conditions  $h_x \neq 0$ ,  $h_{xx} \neq 0$  and substituting the results in successive equations. Doing so we obtain,<sup>5</sup> the following formulas, in which  $a_{i,j;x}$  denote the derivatives of  $a_{i,j}(x)$  and  $A_{ij}$  are integration constants:

$$(1) \ a_{0,0}(x) = A_{00} \cdot h_x^3 \quad \text{from} \quad 2 \cdot h_x \cdot (-h_x \cdot a_{0,0;x} + 3 \cdot a_{0,0}(x) \cdot h_{xx}) = 0;$$

<sup>5</sup>In calculation the authors used Maple<sup>®</sup> software.

- (2)  $a_{1,0}(x) = A_{10} \cdot h_x^3$  from  $2 \cdot h_x \cdot (-h_x \cdot a_{1,0;x} + 3 \cdot h_{xx} \cdot a_{1,0}(x)) = 0$ ;  
(3)  $a_{2,0}(x) = A_{20} \cdot h_x^3$  from  $2 \cdot h_x \cdot (-h_x \cdot a_{2,0;x} + 3 \cdot h_{xx} \cdot a_{2,0}(x)) = 0$ ;  
(4)  $a_{3,0}(x) = A_{30} \cdot h_x^3$  from  $2 \cdot h_x \cdot (-h_x \cdot a_{3,0;x} + 3 \cdot h_{xx} \cdot a_{3,0}(x)) = 0$ ;  
(5)  $a_{3,1}(x) = A_{31} \cdot h_x^2$  from  $2 \cdot h_x \cdot (-h_x \cdot a_{3,1;x} + 2 \cdot a_{3,1}(x) \cdot h_{xx}) = 0$ ;  
(6)  $a_{1,3}(x) = \frac{h_{xx}}{h_x} \cdot a_{0,2}(x)$  from  $2 \cdot h_x \cdot (-h_x \cdot a_{1,3}(x) + h_{xx} \cdot a_{0,2}(x)) = 0$ ;  
(7)  $a_{2,3}(x) = \frac{h_{xx}}{2h_x} \cdot a_{1,2}(x)$  from  $2 \cdot h_x \cdot (-2 \cdot h_x \cdot a_{2,3}(x) + h_{xx} \cdot a_{1,2}(x)) = 0$ ;  
(8)  $a_{3,3}(x) = \frac{h_{xx}}{3h_x} \cdot a_{2,2}(x)$  from  $2 \cdot h_x \cdot (-3 \cdot h_x \cdot a_{3,3}(x) + h_{xx} \cdot a_{2,2}(x)) = 0$ ;  
(9)  $a_{3,2}(x) = 0$  from  $2 \cdot h_x \cdot h_{xx} \cdot a_{3,2}(x) = 0$ ;  
(10)  $a_{0,1}(x) = (-A_{10} \cdot h(x) + A_{01}) \cdot h_x^2$  from  $2 \cdot h_x \cdot (-h_x^4 \cdot A_{10} - h_x \cdot a_{0,1;x} + 2 \cdot a_{0,1}(x) \cdot h_{xx}) = 0$ ;  
(11)  $a_{1,1}(x) = (-2 \cdot A_{20} \cdot h(x) + A_{11}) \cdot h_x^2$  from  $2 \cdot h_x \cdot (-2 \cdot h_x^4 \cdot A_{20} - h_x \cdot a_{1,1;x} + 2 \cdot a_{1,1}(x) \cdot h_{xx}) = 0$ ;  
(12)  $a_{2,1}(x) = (-3 \cdot A_{30} \cdot h(x) + A_{21}) \cdot h_x^2$  from  $2 \cdot h_x \cdot (-3 \cdot h_x^4 \cdot A_{30} - h_x \cdot a_{2,1;x} + 2 \cdot a_{2,1}(x) \cdot h_{xx}) = 0$ ;  
(13)  $a_{2,2}(x) = (-3A_{31} \cdot h(x) + \frac{3}{2}A_{20} \cdot h_x^2 + A_{22}) \cdot h_x$  from  $2 \cdot h_x \cdot (-3 \cdot h_x^3 \cdot A_{31} + 3 \cdot h_x^3 \cdot h_{xx} \cdot A_{20} - h_x \cdot a_{2,2;x} + h_{xx} \cdot a_{2,2}(x)) = 0$ ;  
(\*)  $A_{30} = 0$  from  $6 \cdot h_x^4 \cdot h_{xx} \cdot A_{30} = 0$  at  $y^3 p_x^2 p_y^2$ ;  
(14)  $a_{0,2}(x) = (A_{20} \cdot h(x)^2 - A_{11} \cdot h(x) + \frac{3}{2}A_{00} \cdot h_x^2 + A_{02}) \cdot h_x$  from  $2 \cdot h_x \cdot (2 \cdot h_x^3 \cdot A_{20} \cdot h(x) - h_x^3 \cdot A_{11} + 3 \cdot h_x^3 \cdot h_{xx} \cdot A_{00} - h_x \cdot a_{0,2;x} + h_{xx} \cdot a_{0,2}(x)) = 0$ ;  
(15)  $a_{1,2}(x) = (\frac{3}{2}A_{10} \cdot h_x^2 - 2A_{21} \cdot h(x) + A_{12}) \cdot h_x$  from  $2 \cdot h_x \cdot (2 \cdot h_x^3 \cdot A_{21} - 3 \cdot h_x^3 \cdot h_{xx} \cdot A_{10} + h_x \cdot a_{1,2;x} - h_{xx} \cdot a_{1,2}(x)) = 0$ .

The latter 4 relations yield certain correction in some formulas above:

- (12')  $a_{2,1}(x) = A_{21} \cdot h_x^2$ ;  
(4')  $a_{3,0}(x) = 0$ ;  
(6')  $a_{1,3}(x) = (A_{20} \cdot h(x)^2 - A_{11} \cdot h(x) + \frac{3}{2}A_{00} \cdot h_x^2 + A_{02}) \cdot h_{xx}$ ;  
(7')  $a_{2,3}(x) = (\frac{3}{4}A_{10} \cdot h_x^2 - A_{21} \cdot h(x) + \frac{1}{2}A_{12}) \cdot h_{xx}$ ;  
(8')  $a_{3,3}(x) = (\frac{1}{2}A_{20} \cdot h_x^2 - A_{31} \cdot h(x) + \frac{1}{3}A_{22}) \cdot h_{xx}$ .

Finally, the following formula will be obtained later, we write it here simply for completeness:

$$(16) \quad a_{0,3}(x) = (-A_{10} \cdot h(x) + A_{01}) \cdot h_x^2 - \frac{1}{4}A_{23} \cdot x^2 - \frac{1}{2}A_{24} \cdot x + A_{03}.$$

After calculation of (1)–(15) and (\*) it remains 4 equations. One of them — the coefficient at  $y^0 p_x p_y^3$  — can be written as

$$a_{0,3;x} = 2 \cdot (A_{01} - A_{10} \cdot h(x)) \cdot h_x \cdot h_{xx} - \frac{3}{2} \cdot A_{10} \cdot h_x^3 + (2A_{21} \cdot h(x) - A_{12}) \cdot h_x \quad (5.12)$$

and will be treated later. Three other are the coefficients at  $y^1 p_x p_y^3$ ,  $y^2 p_x p_y^3$  and  $y^3 p_x p_y^3$ . After normalisation we obtain

$$(3A_{00} \cdot h_x^2 + 2A_{20} \cdot h(x)^2 - 2A_{11} \cdot h(x) + 2A_{02}) \cdot h_{xxx} + 6A_{00} \cdot h_{xx}^2 \cdot h_x + 6 \cdot (2A_{20} \cdot h(x) - A_{11}) \cdot h_x \cdot h_{xx} + 6A_{20} \cdot h_x^3 + 4 \cdot (A_{22} - 3A_{31} \cdot h(x)) \cdot h_x = 0 \quad (5.13)$$

$$(3A_{10} \cdot h_x^2 - 4A_{21} \cdot h(x) + 2A_{12}) \cdot h_{xxx} + 6A_{10} \cdot h_{xx}^2 \cdot h_x - 12A_{21} \cdot h_x \cdot h_{xx} = 0 \quad (5.14)$$

$$(3A_{20} \cdot h_x^2 - 6A_{31} \cdot h(x) + 2A_{22}) \cdot h_{xxx} + 6A_{20} \cdot h_{xx}^2 \cdot h_x - 18A_{31} \cdot h_x \cdot h_{xx} = 0 \quad (5.15)$$

Double integration of the latter two gives

$$h_x \cdot (A_{10} \cdot h_x^2 - 4A_{21} \cdot h(x) + 2A_{12}) = A_{23}x + A_{24}, \quad (5.16)$$

$$h_x \cdot (A_{20} \cdot h_x^2 - 6A_{31} \cdot h(x) + 2A_{22}) = A_{33}x + A_{34}, \quad (5.17)$$

where  $A_{23}, A_{24}, A_{33}, A_{34}$  are new integration constants. Substituting the latter relation in the equation (5.12) we can integrate it getting the formula (16) for  $a_{0,3}(x)$  promised above.

The equation (5.13) can not be integrated. However, we can do this even twice with the linear combination “(5.13)” $-2\cdot$ “(5.17)”, and the result is the equation

$$h_x \cdot (A_{00} \cdot h_x^2 + 2A_{20} \cdot h(x)^2 - 2A_{11} \cdot h(x) + 2A_{02}) = -\frac{1}{3}A_{33} \cdot x^3 - A_{34} \cdot x^2 + A_{13} \cdot x + A_{14} \quad (5.18)$$

with new integration constants  $A_{13}, A_{14}$ .

From the above calculation we conclude the following: The metric  $g = h_x^{-2}(dx^2 + dy^2)$  admits a cubic integral of the form (5.11) if and only if the  $h(x)$  satisfies the equations (5.18), (5.16), and (5.17), and then the integral can be constructed using the formulas (1)–(16) for the coefficients  $a_{i,j}(x)$ .

Observe that the non-zero coefficients  $a_{3,j}(x)$  are  $a_{3,1}(x) = A_{31}h_x^2$  and  $a_{3,3}(x) = (\frac{1}{2}A_{20} \cdot h_x^2 - A_{31} \cdot h(x) + \frac{1}{3}A_{22}) \cdot h_{xx}$ . This means that the non-existence of a Jordan block of size  $4 \times 4$  is equivalent to the vanishing of all three coefficients  $A_{20}, A_{22}, A_{31}$ . Suppose for a moment that this is not the case. Then the equation (5.17) is non-trivial. Since we are looking for functions  $h(x)$  for which  $R = h_{xxx}h_x - h_{xx}^2$  is non-constant, Lemma 4.2 says that either  $A_{13}$  or  $A_{14}$  (or both) is non-zero. Consequently, the equation (5.18) is also non-trivial.

We distinguish several possibilities. The first is when  $A_{20} = 0$ . Then integrating (5.17), we obtain

$$-3A_{31} \cdot h(x)^2 + 2A_{22} \cdot h(x) = \frac{1}{2}A_{33}x^2 + A_{34}x + A_{35} \quad (5.19)$$

with an integration constant  $A_{35}$ . We can not have  $A_{31} = 0$  since in this case  $R = h_{xxx}h_x - h_{xx}^2$  would be constant. Also we can not have  $A_{33} = A_{34} = 0$  by the same reason. In the remaining cases we study the behaviour of (branches of) the function  $h(x)$  considered as a solution of an algebraic equation (5.19). It follows that for  $x \rightarrow \infty$  the solution  $h(x)$  grows like  $\sim ax + O(x^0)$  (case  $A_{33} \neq 0$ ) or like  $\sim ax^{1/2} + O(x^0)$  (case  $A_{33} = 0$ ). From (5.19) we obtain the growth of  $h_x$ :  $\sim a + O(x^{-1})$  in the case  $A_{33} \neq 0$  and  $\sim ax^{-1/2} + O(x^{-1})$  in case  $A_{33} = 0$ . In any of these two cases we see that the growth of the l.h.s. of (5.18) is slower than that of the r.h.s. The contradiction shows that we must have  $A_{20} \neq 0$ .

Now subtract the equation (5.17) from (5.18) with coefficient  $\frac{A_{00}}{A_{20}}$ . We obtain the equation

$$h_x \cdot (2A_{20} \cdot h(x)^2 + A'_{11} \cdot h(x) + A'_{02}) = -\frac{1}{3}A_{33} \cdot x^3 - A_{34} \cdot x^2 + A'_{13} \cdot x + A'_{14} \quad (5.20)$$

with new constants  $A'_{11}, A'_{02}, A'_{13}, A'_{14}$  and the same constants  $A_{20} \neq 0, A_{33}, A_{34}$ . Integration gives

$$\frac{2A_{20}}{3} \cdot h(x)^3 + \frac{A'_{11}}{2} \cdot h(x)^2 + A'_{02} \cdot h(x) = -\frac{A_{33}}{12} \cdot x^4 - \frac{A_{34}}{3} \cdot x^3 + \frac{A'_{13}}{2} \cdot x^2 + A'_{14}x + A_{15} \quad (5.21)$$

with an integration constant  $A_{15}$ . Our next possibility is  $A_{33} \neq 0$ . In this case the same argument as above gives the asymptotic  $h(x) \sim ax^{4/3} + O(x)$  and  $h_x \sim \frac{4a}{3}x^{1/3} + O(x^0)$  for  $x \rightarrow \infty$ . Moreover, the coefficient  $a$  satisfies the relation  $A_{33} = -8A_{20} \cdot a^3$ . Substituting these asymptotic in (5.17) gives the asymptotic  $-8A_{31} \cdot a^2 \cdot x^{5/3} + O(x^{4/3})$  for the l.h.s. provided  $A_{31} \neq 0$ . This is a contradiction. Consequently, we must have  $A_{31} = 0$ . But then we obtain the asymptotic  $\frac{64}{27}A_{31} \cdot a^3 \cdot x + O(x^{2/3})$  for the l.h.s., whereas on the r.h.s. we have the leading term  $A_{33}x = -8A_{20} \cdot a^3x$ . As the result we conclude that the case  $A_{33} \neq 0$  is impossible and we must have  $A_{33} = 0$ .

Notice, that in this remaining case  $A_{33} = 0$  we must have  $A_{34}$  since otherwise  $R = h_{xxx}h_x - h_{xx}^2$  would be constant by Lemma 4.2. Here as above from (5.21) we can conclude the asymptotic  $h(x) = ax + O(x^0)$  and  $h_x = a + O(x^{-1})$  for  $x \rightarrow \infty$ . The substitution of these asymptotic in (5.17) gives linear growth  $-6A_{31} \cdot a \cdot x + O(x^0)$  for the l.h.s. provided  $A_{31} \neq 0$ , whereas the r.h.s. is constantly  $A_{34}$ . So we must have  $A_{31} = 0$ . But the only solutions of

the equation (5.17) with  $A_{31} = A_{33} = 0$  are  $h_x = \text{const}$  which contradict the hypothesis  $R = h_{xxx}h_x - h_{xx}^2 \neq \text{const}$ . The latter contradiction demonstrates the non-existence of Jordan blocks of size  $4 \times 4$ .

Finally, we describe cubic integrals  $F$  which give Jordan blocks of size  $3 \times 3$ . The latter condition means that  $F$  is given by (5.11) with vanishing coefficients  $a_{3,j}(x)$ ,  $j = 0, \dots, 3$ . From formulas (1)–(16) we see that this condition is equivalent to vanishing of parameters  $A_{20}, A_{31}, A_{22}$ . In this case we see from (5.17) that the coefficients  $A_{13}, A_{14}$  must also vanish. Thus the equation (5.17) becomes trivial, and the equation (5.18) simplifies to

$$h_x \cdot (A_{00} \cdot h_x^2 - 2A_{11} \cdot h(x) + 2A_{02}) = A_{13} \cdot x + A_{14}. \quad (5.22)$$

The uniqueness of the equation proved in Theorem 4.1 ensures that the equation (5.16) and (5.22) must be proportional to each other and to the equation (1.2) (iii). Let us denote by  $C_1, C_2$  the corresponding proportionality coefficients. This leads to the following relations:

$$\begin{aligned} A_{00} &= C_1 \cdot A_0, & A_{11} &= \frac{1}{2} \cdot C_1 \cdot A_1, & A_{02} &= \frac{1}{2} \cdot C_1 \cdot A_2, & A_{13} &= C_1 \cdot A_3, & A_{14} &= C_1 \cdot A_4, \\ A_{10} &= C_2 \cdot A_0, & A_{21} &= \frac{1}{4} \cdot C_2 \cdot A_1, & A_{12} &= \frac{1}{2} \cdot C_2 \cdot A_2, & A_{23} &= C_2 \cdot A_3, & A_{24} &= C_2 \cdot A_4. \end{aligned} \quad (5.23)$$

Additionally set  $A_{03} = C_{L3}$  and  $A_{01} = \frac{1}{2}C_{LH}$ . Substituting these relations and formulas (1)–(16) we obtain the following formula for a cubic integral  $F$  in the case  $\mu = 0$ :

$$F = C_{L3} \cdot p_y^3 + C_{LH} \cdot \frac{1}{2} h_x^2 (p_x^2 p_y + p_y^3) \quad (5.24)$$

$$\begin{aligned} &+ C_1 \cdot \left( A_0 \cdot h_x^3 \cdot p_x^3 + \frac{y}{2} \cdot A_1 \cdot h_x^2 \cdot p_x^2 p_y + \frac{1}{2} \cdot (3A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2) \cdot h_x \cdot p_x^2 \cdot p_y \right. \\ &\quad \left. \frac{y}{2} \cdot (3A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2) \cdot h_{xx} \cdot p_y^3 \right) \end{aligned} \quad (5.25)$$

$$\begin{aligned} &+ C_2 \cdot \left( A_0 \cdot h_x^3 \cdot y \cdot p_x^3 + \left( \frac{y^2}{4} \cdot A_1 \cdot h_x^2 - A_0 \cdot h_x^2 \cdot h(x) \right) \cdot p_x^2 p_y \right. \\ &\quad \left. + \frac{1}{2} \cdot (3A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2) \cdot h_x \cdot y \cdot p_x p_y^2 \right. \\ &\quad \left. + \left( \frac{1}{4} \cdot (3A_0 \cdot h_x^2 - A_1 \cdot h(x) + A_2) \cdot h_{xx} \cdot y^2 - A_0 \cdot h_x^2 \cdot h(x) - \frac{1}{2} \cdot A_4 \cdot x - \frac{1}{4} \cdot A_3 \cdot x^2 \right) \cdot p_y^3 \right). \end{aligned} \quad (5.26)$$

Thus  $F$  is a linear combination of 4 independent cubic integrals as stated in Theorem 1.1, two of which are  $L^3$  and  $L \cdot H$ , and other two are linear and quadratic in  $y$ .

Notice that the formulas (5.25) and (5.26) for cubic integrals differ from those in (1.5) and (1.6). To obtain the last ones we need to subtract from (5.25) and (5.26) the derivative of the equation (1.2) (iii) with coefficients  $\frac{y}{2} \cdot p_y^3$  and respectively  $\frac{y^2}{4} \cdot p_y^3$ .  $\square$

**5.2. Summing up: Theorem 1.1 is proved.** As we explained in §§2.1, 2.2, 2.3, any metric  $g$  admitting linear integral  $L$  and cubic integral  $F$  such that  $L, H$  and  $F$  are functionally independent have in an appropriate coordinate system the form  $h_x^{-2}(dx^2 + dy^2)$ , where the function  $h(x)$  satisfies (2.8) or (2.13).

In Theorem 4.2 we proved that  $\mu$  in (2.8) is real, or pure imaginary, and the parameters  $A_0, \dots, A_4$  in (2.8) and in (2.13) become real after multiplication with an appropriate constant. Thus, the equations (2.8, 2.13) are essentially the equations (1.2).

It follows from §§2.1, 2.2, 2.3 that the metric  $g = h_x^{-2}(dx^2 + dy^2)$  admits at least one cubic integral  $F_1$  which has the form from Theorem 1.1, and two functionally independent integrals  $F_1, F_2$  in the case  $\mu \neq 0$ . In Section 5 we construct an additional independent cubic integral  $F_2$  in the case  $\mu = 0$ , and explained why there are no other integrals (unless the metric has constant curvature).

Theorem 1.1 is proved.

## 6. GLOBAL SOLUTIONS

In this section we show that if the function  $h(x)$  satisfies the equation (1.2) (ii) and  $h'(x_0) > 0$  at some point  $x_0$  whereas the real parameters  $\mu > 0, A_0, \dots, A_4$  satisfy inequalities  $A_0 > 0, \mu \cdot A_4 > |A_3|$  then the metric  $g = \frac{1}{h_x^2}(dx^2 + dy^2)$  smoothly extends to the sphere  $S^2$  together with the linear integral  $L = p_y$  and the cubic integral  $F$  given by (1.4). More precisely, we show that if  $(r, \varphi)$  are polar coordinates on  $\mathbb{R}^2$  related to  $(x, y)$  by  $r = e^{x/\mu}, \varphi = y/\mu$ , then  $g, L$ , and  $F$  are well-defined on the punctured plane  $\mathbb{R}^2 \setminus \{0\}$  and extend smoothly to the origin  $0$  and to the infinity point  $\infty$  of the Riemann compactification  $S^2 = \overline{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$ , such that the extended tensor  $g$  is still a (non-degenerate) Riemannian metric on the whole sphere  $S^2$ .

The obtained in this way family of examples of superintegrable metrics on the sphere is new. Indeed, by [Ki] Darboux-superintegrable metrics can not live on a closed manifold, so the only known superintegrable metrics on the 2-sphere are the standard metrics of constant curvature. In view of Corollary 3.1, for most values of the parameters satisfying the above conditions, the metrics are not metrics of constant curvature.

Let us describe the conditions on parameters  $\mu > 0, A_0, \dots, A_4$  which distinguish our global solutions. Since we want that the linear integral  $L$  to be also globally defined on  $S^2$ , the Killing vector field must be as in the standard rotation (see for example [Bo-Ma-Fo]), and we must have the elliptic case, i.e.,  $h$  must satisfy (1.2), (ii). Since Darboux-superintegrable case is impossible on closed surfaces due to [Ki],  $A_0$  is non-zero. Then dividing the equation by  $A_0$  we obtain  $A_0 = 1$ . Applying the action  $h(x) \mapsto h(x) + c$  we can make  $A_1 = 0$ . Since  $\mu \neq 0$ , making an appropriate rescaling in  $x$  we can make  $\mu = 1$ . Further, we rewrite the free term  $A_3 \sinh(x) + A_4 \cosh(x)$  in the form  $A_+ e^x + A_- e^{-x}$  and we impose the positivity condition  $A_+, A_- > 0$ . This means that the free term is positive for all  $x \in \mathbb{R}$ . An appropriate translations in  $x$  direction transforms the term  $A_+ e^x + A_- e^{-x}$  into  $A_e \cdot (e^x + e^{-x})$ . It is easy to see that in terms of the original parameters our conditions are

$$\mu > 0, \quad A_0 > 0, \quad A_4 \cdot \mu - |A_3| > 0. \quad (6.1)$$

**Lemma 6.1.** *For any  $A_e > 0, A_2$  and  $h(x_0)$  there exists a unique real-analytic local solution  $h(x)$  of the ODE*

$$\mathcal{E} := h_x(h_x^2 - h(x)^2 + A_2) - A_e \cdot (e^x + e^{-x}) = 0 \quad (6.2)$$

*with the initial value  $h(x_0)$  such that  $h_x(x_0)$  is the unique positive root of the characteristic polynomial  $\chi(\lambda) := \lambda(\lambda^2 + A_2 - h(x_0)^2) - A_e \cdot (e^{x_0} + e^{-x_0})$ .*

*Proof.* Considering the graph of the polynomial  $\lambda(\lambda^2 + a)$  for different  $a$  (mainly, for the cases  $a \geq 0$  and  $a < 0$ ) we see that there exists a unique positive solution of the equation  $\lambda(\lambda^2 + a) = A$  with  $A > 0$  which depends real-analytically on  $a$  and  $A > 0$ . The standard theory of ODEs implies now the desired local existence and uniqueness of solutions of (6.2).  $\square$

**Remark.** Notice that depending on coefficients  $A > 0$  and  $a$  one could have two distinct, one double, or none *negative* roots of the equation  $\lambda(\lambda^2 + a) = A$ . Vice versa, in the case  $A < 0$  there is one negative root and there could be two distinct, one double, or none positive roots. This explains our ‘‘Ansatz’’: we need that the right hand side  $A_3 \sinh(x) + A_4 \cosh(x) = A_+ e^x + A_- e^{-x}$  remains positive.

Let  $h(x)$  be a local solution constructed in the previous lemma. Make the substitution  $x = \log(t)$ . Then the equation (6.2) transforms into

$$\tilde{\mathcal{E}} := t \cdot h_t((t \cdot h_t)^2 - h(t)^2 + A_2) - A_e \cdot (t + t^{-1}) = 0 \quad (6.3)$$

defined for all  $t > 0$ .

**Proposition 6.1.** *Any local solution  $h(t)$  of (6.3) with  $h_t$  positive extends to the interval  $t \in (0, +\infty)$ . Moreover, the functions  $t \cdot h(t)$ ,  $t^2 h_t$ , and  $t^3 h_{tt}$  extend real-analytically to some interval  $t \in (-\varepsilon, \varepsilon)$ ,  $\varepsilon > 0$ . In particular, the function  $h(t)$  has a simple pole at  $t = 0$ .*

Notice that in terms of the variable  $x$  the first assertion of the proposition states that every local solution  $h(x)$  as in Lemma 6.1 is global, i.e., is defined for all  $x \in \mathbb{R}$ .

*Proof.* Considering the local behaviour of  $h(t)$  at  $t = 0$ , we show a stronger property, namely, that  $h(t) = f(t^2)/t$  for some real-analytic function  $f(\tau)$ , defined in some interval  $\tau \in [-\varepsilon, +\infty]$ . This means that we make the substitution  $t = \sqrt{\tau}$ , or equivalently,  $t^2 = \tau$  or  $x = \frac{1}{2} \log \tau$ . However, we consider all three functions  $h(x)$ ,  $h(t)$ , and  $h(\tau)$  as a single object given in different coordinates.

For  $A$  positive and  $h, A_2$  real, denote by  $\eta = \eta(h, A_2; A)$  the unique positive root of the polynomial  $\lambda(\lambda^2 + A_2 - h^2) - A$ . Then  $\eta(h, A_2; A)$  is monotone in every its argument  $A > 0$ ,  $A_2$  and  $h \neq 0$ :  $\eta(h, A_2; A)$  will increase if we increase  $A > 0$  and the absolute value  $|h|$  and decrease  $A_2$ . Consequently, for any given  $A_2$  and  $A_e > 0$  any solution  $h(x)$  of (6.2) with positive  $h_x$  satisfies  $h_x \geq \eta^* := \eta(0, A_2; 2A_e)$ . In particular,  $h(x)$  is monotone.

Now let us observe the following two facts. First, the translation in  $x$  (which means the multiplication of  $t$  by a constant) transforms our equation (6.2) into

$$h_x(h_x^2 - h(x)^2 + A_2) - A_+ \cdot e^x - A_- \cdot e^{-x} = 0 \quad (6.4)$$

or respectively (6.3) into

$$t \cdot h_t((t \cdot h_t)^2 - h(t)^2 + A_2) - A_+ \cdot t - A_- \cdot t^{-1} = 0 \quad (6.5)$$

with positive parameters  $A_+, A_-$ . Second, for any values of parameters our equation has a very simple function as the solution. Namely, the function  $\tilde{h}(t) := C_h(-A_-/t + A_+ \cdot t)$  satisfies the equation (6.5) if and only if the constant  $C_h$  is the unique positive root of the polynomial  $C_h(4A_+A_- \cdot C_h^2 + A_2) = 1$ . We use these special solutions and their translations in  $x$  ( $\Leftrightarrow$  reparametrisations  $t \mapsto c \cdot t$ ) to estimate the behaviour of a general solution  $h(t)$ .

Let  $h(t)$  be any solution of (6.3) defined in a neighbourhood of the initial value  $t_0$ . Denote  $h_0 := h|_{t=t_0}$ . Let  $\tilde{h}(t)$  be the unique solution of (6.5) of the form  $\tilde{h}(t) := C_h(-A_-/t + A_+ \cdot t)$  with  $C_h > 0$ . In the case when  $\tilde{h}(t_0) = h_0$  we must have  $h(t) = \tilde{h}(t)$  everywhere and thus  $h(t)$  is defined globally on  $(0, +\infty)$ .

Assume that  $\tilde{h}(t_0) > h_0$ . Since both functions are solution of the same 1st order ODE,  $\tilde{h}(t) > h(t)$  for every  $t$  from the maximal existence interval  $(t_-, t_+) \subset (0, +\infty)$  of the solution  $h(t)$ . Further, since  $h(t)$  is monotone increasing and bounded from above by the function  $\tilde{h}(t)$  defined on the whole ray  $(0, +\infty)$ , we conclude that the solution  $h(t)$  does not explode and exists on the whole interval  $(t_0, +\infty)$ . In particular, the existence interval for  $h(t)$  is  $(t_-, +\infty)$ . Moreover,  $h(t) \leq C_h A_+ t$  for all  $t \in (t_-, +\infty)$ .

Our next step is construction of a similar lower bounding function  $h_-(t)$  which also has the form  $h_-(t) = C_h \cdot (-A'_-/t + A_+ \cdot t)$  and satisfies the equation (6.5) with some new parameters  $A'_-$  and  $A_2$ . This means that constructing  $h_-(t)$  from  $\tilde{h}(t)$  we change only one parameter,

namely  $A_-$ . Increasing it the value  $h_-(t_0)$  will decrease. Thus we can find the new value  $A'_- > A_-$  from the condition  $h_-(t_0) = h_0$ . Then we find the new value  $A'_2$  from the relation  $C_h(4A_+A'_- \cdot C_h^2 + A'_2) = 1$ . Notice that since  $A'_-$  was increased,  $A'_2$  is lower than  $A_2$ , i.e.,  $A'_2 < A_2$ . Assume additionally that  $h_0 \leq 0$ . Then  $h(t) < 0$  for every  $t \in (t_0, t_0)$ . Now using the monotonicity on the function  $\eta(h, A_2; A)$  and comparing the equations for  $h(t)$  and  $h_-(t)$  and the initial values  $h_-(t_0) = h(t_0)$  and  $\frac{d}{dt}h_-(t_0) > \frac{d}{dt}h(t_0)$ , we conclude the following: for every  $t$  in the whole existence interval  $(t_-, t_0)$  left from the initial point  $t_0$  we have  $h_-(t) < h(t)$ ,  $|h_-(t)|^2 > |h(t)|^2$ , and  $\frac{d}{dt}h_-(t) > \frac{d}{dt}h(t)$ .

The monotonocity argument above can be applied in the case when  $h_0 > 0$ . In this case we need an additional step. Namely, since we have the uniform estimate  $h'(t) \geq \eta^* > 0$ , our solution  $h(t)$  vanishes at the unique  $t_1$  lying in the existence interval  $(t_-, t_0)$  left from  $t_0$ . Then we apply the same argument above at the point  $t_1$  instead of  $t_0$ .

Finally, recall that above we have proceeded under assumption  $\tilde{h}(t_0) > h_0$ . The remaining case  $\tilde{h}(t_0) < h_0$  is treated quite similarly, and we only indicates the changes. In this new situation the function  $\tilde{h}(t) := C_h(-A_-/t + A_+ \cdot t)$  will be a lower bound for our solution  $h(t)$ , i.e.,  $h(t) > \tilde{h}(t)$  everywhere in the existence interval. This fact together with monotonicity of  $h(t)$  will imply the extensibility of  $h(t)$  left until the value  $t_- = 0$ . To construct the upper bounding function  $h^+(t)$  we increase the parameter  $A_+$  and also decrease  $A_2$ . Moreover, in the case  $h_0 < 0$  we need an additional step, in which we go right to the unique point  $t_1 \in (t_0, t_+)$  such that  $h(t_1) = 0$ , and apply the argument at this point  $t_1$ .

For the second assertion of the proposition in the case  $\tilde{h}(t_0) < h_0$  we need one bounding function, namely, the function  $h_+(t)$  of the familiar form  $h_+(t) := C_h(-A'_-/t + A_+ \cdot t)$  such that  $h_+(t) > h(t)$  for  $t < t_0$  ( $h^+(t)$  does this for  $t > t_0$  or respectively for  $t > t_1 > t_0$ ). As before in the case  $h_0 > 0$  we go left to some point  $t_2 < t_0$  such that  $h(t_2) < 0$ . Notice that  $\tilde{h}(t_2)$  is still less than  $h(t_2)$ , and hence  $\tilde{h}(t_2) < 0$ . Now, if we start to decrease  $A_-$ , the value  $\tilde{h}(t_2)$  will increase until it arrives to 0. Consequently, for some  $A'_0 \in (0, A_0)$  the function  $h_+(t) := C_h(-A'_0/t + A_+ \cdot t)$  has value  $h_+(t_2) = h(t_2)$ . The same monotonicity argument as above gives us  $h(t) < h_+(t)$  for  $t \in (0, t_2)$ .

This gives the desired global existence of the solution  $h$ : the maximal existence interval is  $(0, +\infty)$  for the coordinate  $t$  which means that  $h(x)$  exists for all  $x \in \mathbb{R}$ .

As a consequence of the above argument, we obtain the estimate  $-\frac{c}{t} < h(t) < -\frac{c}{t}$  for  $0 < t \ll 1$  with some positive constants  $c, C$  for any solution of (6.3).

Now let us make the substitution  $t = \sqrt{\tau}$  and  $h(t) = f(t^2)/t = f(\tau)/\sqrt{\tau}$ . Then the equation (6.3) transforms into

$$8 \cdot f_\tau^3 \cdot \tau^2 + (-12 \cdot f_\tau^2 \cdot f(\tau) - A_e + 2 \cdot f_\tau \cdot A_2) \cdot \tau - f(\tau) \cdot A_2 - A_e + 4 \cdot f(\tau)^2 \cdot f_\tau = 0 \quad (6.6)$$

For  $\tau$  small enough and  $f < 0$  we can resolve this equation with respect to the derivative  $f_\tau$ , and then (6.6) transforms into the form

$$f_\tau = \Phi(\tau, f(\tau); A_2, A_e) \quad (6.7)$$

for some real-analytic function  $\Phi(\tau, f; A_2, A_e)$  of arguments  $\tau \in [-\varepsilon, \varepsilon]$ ,  $f < 0$ ,  $A_2, A_e$ . In particular,  $\Phi|_{\tau=0} = \frac{A_e + A_2 \cdot f}{4f^2}$ . It follows, that for every  $A_2, A_e$ , every negative  $f_0$ , and every  $\tau_0$  small enough there exists the unique solution of the equation (6.6) with the initial value  $f(\tau_0) = f_0$ . More precisely, for given intervals  $A_2, A_e \in [-C, C]$ ,  $f_0 \in [-C, -c]$  with  $0 < c < C$  there exists an  $\varepsilon = \varepsilon(c, C) > 0$  such that the equation (6.6) has the unique

solution with the initial value  $f(\tau_0) = f_0$  at  $\tau_0 \in [-\varepsilon, \varepsilon]$  such that  $f_\tau(\tau_0) = \Phi(\tau_0, f_0; A_2, A_e)$ . More over, this solution is well-defined and real-analytic on the whole interval  $\tau \in [-\varepsilon, \varepsilon]$ .

Next, we observe that the above estimate  $-\frac{c}{t} < h(t) < -\frac{c}{t}$  for  $0 < t \ll 1$  is equivalent to the estimate  $-C < f(\tau) < -c$  for  $0 < \tau \ll 1$ . The proposition follows.  $\square$

Now we state the result about the extensibility of  $g$  and  $F$  to a metric and a cubic integral defined globally on  $S^2$ .

**Theorem 6.1.** *For any  $A_e > 0, A_2$  and  $h_0$ , let  $h(t)$  be the unique solution of the equation (6.3) with the initial value  $h|_{t=1} = h_0$  and with  $h_t|_{t=1} > 0$ . Then the metric*

$$g = \frac{dt^2 + t^2 \cdot d\varphi^2}{t^4 h_t^2} \quad (6.8)$$

*defined on the plane  $\mathbb{R}^2$  with the polar coordinates  $(t, \varphi)$  extends to a real analytic metric on the sphere  $S^2 = \overline{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$  with the Killing vector  $\mathbf{v} = \frac{\partial}{\partial \varphi}$  which admits a cubic integral  $F$  also well-defined and real-analytic globally on  $S^2$ .*

*Moreover, the metric  $g$  has constant curvature  $\kappa$  if and only if  $h_0 = 0$ .*

*Proof.* By Proposition 6.1 the function  $h(t)$  is well-defined for all  $t \in (0, +\infty)$  and for  $t$  small enough  $h(t) = \frac{f(t^2)}{t}$  with some real-analytic function  $f(\tau)$  such that  $f(0) < 0$ . Consequently, in a neighbourhood of the origin the function  $t^2 h_t$  is real-analytic and non-vanishing. It follows that the formula  $g = \frac{dt^2 + t^2 \cdot d\varphi^2}{t^4 h_t^2}$  defines a non-degenerate real-analytic Riemannian metric in a neighbourhood of the origin in  $\mathbb{R}^2$  with the polar coordinates  $(t, \varphi)$ . Substitution  $t = e^x, \varphi = y$  transforms this metric into the familiar form  $g = \frac{dx^2 + dy^2}{h_x^2}$  and the equation (6.3) into (6.2). By Proposition 6.1 the metric  $g = \frac{dx^2 + dy^2}{h_x^2}$  is well-defined for all  $x \in \mathbb{R}$  or equivalently for all  $t \in (0, +\infty)$ . This means that the metric  $g = \frac{dt^2 + t^2 \cdot d\varphi^2}{t^4 h_t^2}$  is well-defined on the whole plane  $\mathbb{R}^2$ .

To show the extensibility to the infinity point  $\infty$  we apply the inversion of the sphere  $S^2 = \mathbb{R}^2 \cup \{\infty\}$  with respect to the unit circle given by the condition  $t = 1$ . Recall that the inversion map interchanges the origin 0 and the infinity point  $\infty$  and that in the polar coordinates it is given by  $(t, \varphi) \mapsto (t^{-1}, \varphi)$ . Changing to the coordinates  $x = \log(t), y = \varphi$  we obtain the formula  $(x, y) \mapsto (-x, y)$ . So we conclude immediately that the extensibility of the metric  $g$  to the infinity  $\infty$  is equivalent to the extensibility to the origin 0, and thus this is the case.

Let  $\xi := t \cdot \cos(\varphi)$  and  $\eta := t \cdot \sin(\varphi)$  be the Cartesian coordinates corresponding to the polar coordinates  $(t, \varphi)$  and  $p_\xi, p_\eta$  the corresponding momenta, i.e., dual coordinates on  $T^*S^2$ . Then the vector field  $\partial_y = \partial_\varphi$  is given by  $\xi \partial_\eta - \eta \partial_\xi$ . This means that the linear integral  $L = p_y$  is given by  $L = \xi p_\eta - \eta p_\xi$  and hence  $L$  extends smoothly to the origin. By the symmetry argument  $L$  extends also to the infinity point  $\infty$ .

It remains to show that the cubic integral  $F$  given by (1.4) also extends to the origin and to the infinity point  $\infty$ . Our argumentation is as follows. First, we substitute in the formulas (1.4) our values of parameters  $A_0 = \mu = 1$  and  $A_1 = 0$ . Next, without loss of generality we may set  $\phi = 0$  for ‘‘phase parameter’’ in (1.4). Then each function  $a_0(x), \dots, a_3(x)$  in (1.4) becomes a sum of linear and cubic monomials in functions  $h(x), h_x, h_{xx}$ . Now we observe that each function  $h(x), h_x, h_{xx}$ , considered as a function of the variable  $t$ , has the form  $f_i(t^2)/t$  for certain real-analytic function  $f_i(\tau)$ , namely,  $f_0(\tau) = f(\tau)$  as in the proof of Proposition 6.1,  $f_1(\tau) = 2\tau f'(\tau) - f(\tau)$  and  $f_2(\tau) = 4\tau^2 f''(\tau) + f(\tau)$ . Further, in the same way as it

was done for  $p_y = \xi \partial_\eta - \eta \partial_\xi$ , we obtain the formula  $p_x = \xi p_\xi + \eta p_\eta$ . Substituting all these relations and also the relations  $\cos(y) = \frac{\xi}{t}$ ,  $\sin(y) = \frac{\eta}{t}$  in (1.4) we see that in coordinates  $(\xi, \eta)$  the integral  $F$  has the form

$$F(\xi, \eta; p_\xi, p_\eta) = \sum_{i=0}^3 \sum_{j=0}^4 p_\xi^{3-i} p_\eta^i \xi^{4-j} \eta^j \frac{\psi_{ij}(\tau)}{\tau^2}$$

where  $\tau = t^2 = \xi^2 + \eta^2$  and  $\psi_{ij}(\tau)$  are some real-analytic functions. It follows that to establish the real-analyticity of  $F$  we need to know only the linear parts of the functions  $\psi_{ij}(\tau)$ . Considering the expressions of these linear parts of  $\psi_{ij}(\tau)$  in terms of the functions  $h(x), h_x, h_{xx}$  we see that only two lower monomials of each function  $h(x), h_x, h_{xx}$  are involved. We write  $h(t) = -c_0 \cdot t^{-1} + c_1 \cdot t + O(t^3)$ . Differentiation yields  $h_x = t \cdot h_t = +c_0 \cdot t^{-1} + c_1 \cdot t + O(t^3)$  and  $h_{xx} = t \cdot \frac{d}{dt}(t \cdot h_t) = -c_0 \cdot t^{-1} + c_1 \cdot t + O(t^3)$ . Substituting these expressions in the equation (6.3) and considering the coefficients by  $t^{\pm 1}$ , we obtain algebraic equations

$$4 \cdot c_0^2 \cdot c_1 + c_0 \cdot A_2 - A_e = 0, \quad 4 \cdot c_1^2 \cdot c_0 + c_1 \cdot A_2 - A_e = 0.$$

So we conclude the equality  $c_1 = c_0$  and the formula  $A_e = c_0 \cdot (4 \cdot c_0^2 + A_2)$ . Now, substituting all these formulas in (1.4) we obtain

$$\begin{aligned} F &= c_0^3 \cdot p_\xi \cdot (p_\xi^2 + p_\eta^2) + (3 \cdot c_0^3 \cdot \xi^2 + \frac{c_0 \cdot (A_2 + 6 \cdot c_0^2)}{2} \cdot \eta^2) \cdot p_\xi^3 - c_0 \cdot (2 \cdot c_0^2 + A_2) \cdot \eta \cdot \xi \cdot p_\xi^2 \cdot p_\eta \\ &+ (\frac{c_0 \cdot (10 \cdot c_0^2 + A_2)}{2} \cdot \xi^2 + c_0^3 \cdot \eta^2) \cdot p_\eta^2 \cdot p_\xi + 2 \cdot c_0^3 \cdot \eta \cdot \xi \cdot p_\eta^3 + O(\xi^4 + \eta^4) \end{aligned}$$

in which  $O(\xi^4 + \eta^4)$  means term of higher degree in  $\xi, \eta$ .

This shows that the cubic integral  $F$  extends real-analytically to the origin as desired. The extensibility of  $F$  to the infinity point  $\infty$  can be obtained from the extensibility to the origin by means of the inversion.

The theorem follows.  $\square$

## 7. CONCLUSION

We found all two-dimensional Riemannian metrics whose geodesic flows admit one integral linear in momenta ( $L$ ) and one integral cubic in momenta ( $F$ ) such that  $L, F$  and the Hamiltonian  $H$  of the geodesic flow are functionally independent. Within these metrics, we point out the metrics that are already known, and proved that most of our metrics are new. We have also showed that, in the case when the parameters satisfy certain inequalities, the metric and the integrals  $L$  and  $F$  extend real-analytically to the sphere  $S^2$ , giving new unexpected examples of integrable metrics on the sphere.

The results and the methods of our paper suggest the following directions of further investigations.

**Problem 1.** *Generalise our result for integrals of higher degree.*

In other words, we suggest to construct all two-dimensional metrics whose geodesic flows admit one integral linear in momenta ( $L$ ) and one integral polynomial in momenta of degree 4, (5, 6, etc.) in momenta ( $F$ ) such that  $L, F$  and  $H$  are functionally independent.

The main trick that allowed us to solve the case (linear integral  $L$  + cubic integral  $F$ ) survives in this setup: the Poisson bracket  $\{L, F\}$  is again an integral of the same degree as  $F$ . Arguing as in §2.1 one can reduce the problem to analyse of certain system of ODE. Though it is not clear in advance whether one can reduce this system of ODE to one equation (as we did in the case (linear integral  $L$  + cubic integral  $F$ )), the approach should at least allow to construct new examples of superintegrable metrics.

**Problem 2.** *Generalize our results for pseudo-Riemannian metrics.*

We expect that this is possible to do the local description using the same idea. We do not expect that one can find the analog of our global examples on closed surfaces in the pseudo-Riemannian case. Generally, it could be complicated to generalize global Riemannian construction to the pseudo-Riemannian setting. In certain cases though the existence of additional structure such as additional integrals (as for example in [Ma3]) allows to keep control over the situation.

**Problem 3.** *Quantise the cubic integral.*

Take a metric  $g$  from Theorem 1.1 and consider its Laplacian  $\Delta$  (since our metric is  $\frac{1}{h_x^2}(dx^2 + dy^2)$ ,  $\Delta = h_x^2 \left( \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \right)$ ; one can view it as a mapping  $\Delta : C^\infty(M^2) \rightarrow C^\infty(M^2)$  though one also can consider Laplacian as a linear operator on bigger function spaces).

Does there exist a differential operator  $\tilde{F}$  of degree 3,

$$\begin{aligned} \tilde{F} &= a_0(x, y) \frac{\partial^3}{\partial^3 x} + a_1(x, y) \frac{\partial^2}{\partial^2 x} \frac{\partial}{\partial y} + a_2(x, y) \frac{\partial}{\partial x} \frac{\partial^2}{\partial^2 y} + a_3(x, y) \frac{\partial^3}{\partial^3 y} \\ &+ b_0(x, y) \frac{\partial^2}{\partial^2 x} + b_1(x, y) \frac{\partial}{\partial x} \frac{\partial}{\partial y} + b_2(x, y) \frac{\partial^2}{\partial^2 y} + c_0(x, y) \frac{\partial}{\partial x} + c_1(x, y) \frac{\partial}{\partial y} + d(x, y) \end{aligned}$$

such that  $\tilde{F}$  commute with  $\Delta$ , i.e., such that for every smooth function  $f : M^2 \rightarrow \mathbb{R}$

$$[\Delta, \tilde{F}](f) := \Delta(\tilde{F}(f)) - \tilde{F}(\Delta(f)) \equiv 0,$$

and such that its symbol

$$a_0(x, y)p_x^3 + a_1(x, y)p_x^2 p_y + a_2(x, y)p_x p_y^2 + a_3(x, y)p_y^3$$

coincides with the integral  $F$  from Theorem 1.1?

Recall that for all previously known superintegrable systems such quantisation of the integral was possible, and was extremely useful for the describing of eigenfunctions of  $\Delta$ . Note that quantum superintegrability can be most effectively used (if it exists) in the case of metrics from Theorem 6.1, since in this case the Laplacian is a selfadjoint operator.

**Problem 4.** *Find physical or mechanical systems realizing the Hamiltonian systems corresponding to (at least some) metrics constructed in Theorem 1.1.*

This problem is an interesting challenge for both mathematicians and physicists, especially in the case of global systems given by Theorem 6.1. Let us note that many classical examples of global integrable systems have arisen as the mathematical models for concrete naturally defined dynamical systems in physics and mechanics, and that many superintegrable metrics have physical realisation or can be applied to solving of physical problems.

**Problem 5.** *Describe the metrics from Theorem 6.1 in the terms of [Be, Chapter 4].*

As we already mentioned above, all geodesics of the metrics from Theorem 6.1 are closed. By construction, the metrics are the metrics of revolution. Then, these metrics are a subclass of the so-called Tannery metrics from [Be, Chapter 4].

**Problem 6.** *Find isometric imbeddings of metrics from Theorem 6.1 in  $(\mathbb{R}^3, g_{standard})$ .*

Such isometric imbeddings are possible at least for certain metrics from 6.1, since they have positive curvature as small perturbations of the standard metric. This problem is related to Problem 4, and is a geometric analog of it. It also is related to Problem 5 in view of [Be, Chapter 4(C)].

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