TWO REMARKS ON PQ^{ϵ}-PROJECTIVITY OF RIEMANNIAN METRICS

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Abstract. We show that PQ^{ϵ}-projectivity of two Riemannian metrics introduced in [15] implies affine equivalence of the metrics unless $\epsilon \in \{0, -1, -3, -5, -7, ...\}$. Moreover, we show that for $\epsilon = 0$, PQ^{ϵ}-projectivity implies projective equivalence.

1. Introduction

1.1. PQ^{ϵ}-projectivity of Riemannian metrics. Let $g, \bar{g}$ be two Riemannian metrics on an m-dimensional manifold M. Consider $(1,1)$-tensors P, Q which satisfy

$$\begin{gathered} g(P.,.) = -g(.,P.), \;\; g(Q.,.) = -g(.,Q.) \\ \bar{g}(P.,.) = -\bar{g}(.,P.), \;\; \bar{g}(Q.,.) = -\bar{g}(.,Q.) \\ PQ = \epsilon Id, \end{gathered} \tag{1}$$

where Id is the identity on TM and ϵ is a real number, $\epsilon \neq 1, m+1$. The following definition was introduced in [15].

Definition 1. The metrics $g, \bar{g}$ are called PQ^{ϵ}-projective if for a certain 1-form Φ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \tag{2}$$

for all vector fields X, Y.

Example 1. If the two metrics g and $\bar{g}$ are *affinely equivalent*, i.e. $\nabla = \bar{\nabla}$, then they are PQ^{ϵ}-projective with P, Q, ϵ arbitrary and $\Phi \equiv 0$.

Example 2. Suppose that $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. It follows that equation (2) becomes

$$\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X. \tag{3}$$

By Levi-Civita [4], equation (3) is equivalent to the condition that g and $\bar{g}$ have the same geodesics considered as unparametrized curves, i.e., that g and $\bar{g}$ are *projectively equivalent*. The theory of projectively equivalent metrics has a very long tradition in differential geometry, see for example [13, 10, 7, 5, 6] and the references therein.

Example 3. Suppose that $P = Q = J$ and $\epsilon = -1$. It follows that J is an almost complex structure, i.e., $J^2 = -Id$, and by (1) the metrics g and $\bar{g}$ are required to be hermitian with respect to J. Equation (2) now reads

$$\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX. \tag{4}$$

This equation defines the *h-projective equivalence* of the hermitian metrics g and $\bar{g}$ and was introduced for the first time by Otsuki and Tashiro in [12, 14] for Kählerian metrics. The theory of h-projectively equivalent metrics was introduced as an analog of projective geometry in the Kählerian situation and has been studied actively over the years, see for example [11, 3, 1, 2, 8] and the references therein.

Remark 1. PQ^{ϵ}-projectivity of Riemannian metrics is a special case of so-called F-planar mappings introduced and investigated in [9], whose defining equation [9, (1)] clearly generalises equation (2) above.

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1.2. **Results.** The aim of our paper is to give a proof of the following two theorems:

Theorem 1. *Let Riemannian metrics g and $\bar g$ be PQ^ϵ-projective. If g and $\bar g$ are not affinely equivalent, the number ϵ is either zero or an odd negative integer, i.e., $\epsilon \in \{0, -1, -3, -5, -7, ...\}$.*

Theorem 2. *Let Riemannian metrics g and $\bar g$ be PQ^ϵ-projective. If $\epsilon = 0$ then g and $\bar g$ are projectively equivalent.*

1.3. **Motivation and open questions.** As it was shown in [15], PQ^ϵ-projectivity of the metrics $g, \bar g$ allows us to construct a family of commuting integrals for the geodesic flow of g (see Fact 2 and equation (9) below). The existence of these integrals is an interesting phenomenon on its own. Besides, it appeared to be a powerful tool in the study of projectively equivalent and h-projectively equivalent metrics (Examples 2,3), see [3, 7, 5, 6, 8]. Moreover, in [15] it was shown that given one pair of PQ^ϵ-projective metrics, one can construct an infinite family of PQ^ϵ-projective metrics. Under some non-degeneracy condition, this gives rise to an infinite family of integrable flows.

From the other side, the theories of projectively equivalent and h-projectively equivalent metrics appeared to be very useful mathematical theories of deep interest.

The results in our paper suggest to look for other examples in the case when $\epsilon = -1, -3, -5, ...$ If $\epsilon = -1$ but $P^2 \neq -Id$, a lot of examples can be constructed using the "hierarchy construction" from [15]. It is interesting to ask whether every pair of PQ^{-1}-projective metrics is in the hierarchy of some h-projectively equivalent metrics.

Another attractive problem is to find interesting examples for $\epsilon = -3, -5, ...$ Besides the relation to integrable systems provided by [15], one could find other branches of differential geometry of similar interest as projective or h-projective geometry.

1.4. **PDE for PQ^ϵ-projectivity.** Given a pair of Riemannian metrics $g, \bar g$ and tensors P, Q satisfying (1), we introduce the $(1,1)$-tensor $A = A(g, \bar g)$ defined by

$$A = \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{m+1-\epsilon}} \bar g^{-1} g. \tag{5}$$

Here we view the metrics as vector bundle isomorphisms $g : TM \to T^*M$ and $\bar g^{-1} : T^*M \to TM$. We see that A is non-degenerate and self-adjoint with respect to g and $\bar g$. Moreover A commutes with P and Q.

Fact 1 (Lemma 2 in [15], see also Theorems 5,6 in [9])**.** *Two metrics g and $\bar g$ are PQ^ϵ-projective if for a certain vector field Λ, the $(1,1)$-tensor A defined in (5) is a solution of*

$$(\nabla_X A)Y = g(Y, X)\Lambda + g(Y, \Lambda)X + g(Y, QX)P\Lambda + g(Y, P\Lambda)QX \text{ for all } X, Y \in TM. \tag{6}$$

Conversely, if A is a g-self-adjoint positive solution of (6) which commutes with P and Q, the Riemannian metric

$$\bar g = (\det A)^{-\frac{1}{1-\epsilon}} g A^{-1}$$

is PQ^ϵ-projective to g.

Remark 2. Taking the trace of the $(1,1)$-tensors in equation (6) acting on the vector field Y, we obtain

$$\Lambda = \frac{1}{2(1-\epsilon)} \operatorname{grad} \operatorname{trace} A, \tag{7}$$

hence, (6) is a linear first order PDE on the $(1,1)$-tensor A.

Remark 3. From Fact 1 it follows that the metrics $g, \bar g$ are affinely equivalent if and only if $\Lambda \equiv 0$ on the whole M.

Remark 4. The relation between the 1-form Φ in (2) and the vector field Λ in (6) is given by $\Lambda = -Ag^{-1}\Phi$ (again $g^{-1} : T^*M \to TM$ is considered as a bundle isomorphism), see [15]. Recall from Example 2 that projective equivalence is a special case of PQ^ϵ-projectivity with $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. In view of Fact 1, we now have that g and $\bar g$ are projectively equivalent if and only if $A = A(g, \bar g)$ given by (5) (with $\epsilon = 0$), satisfies (6) with $P\Lambda = 0$ or $Q = 0$, i.e.,

$$(\nabla_X A)Y = g(Y, X)\Lambda + g(Y, \Lambda)X \text{ for all } X, Y \in TM. \tag{8}$$

2. Proof of the results

2.1. **Topalov's integrals.** We first recall

Fact 2 (Proposition 3 in [15]). *Let g and $\bar g$ be PQ^ϵ-projective metrics and let A be defined by* (5). *We identify TM with T^*M by g, and consider the canonical symplectic structure on $TM \cong T^*M$. Then the functions $F_t : TM \to \mathbb{R}$,*

$$F_t(X) = |\det(A - tId)|^{\frac{1}{1-\epsilon}} g((A - tId)^{-1}X, X), \quad X \in TM \tag{9}$$

are commuting quadratic integrals for the geodesic flow of g.

Remark 5. Note that the function F_t in equation (9) is not defined in the points $x \in M$ such that $t \in \operatorname{spec} A_{|x}$. From the proof of Theorem 1 it will be clear that in the non-trivial case one can extend the functions F_t to these points as well.

2.2. **Proof of Theorem 1.** Suppose that g and $\bar g$ are PQ^ϵ-projective Riemannian metrics and let $A = A(g, \bar g)$ be the corresponding solution of (6) defined by (5). Since A is self-adjoint with respect to the positively-definite metric g, the eigenvalues of A in every point $x \in M$ are real numbers. We denote them by $\mu_1(x) \le ... \le \mu_m(x)$; depending on the multiplicity, some of the eigenvalues might coincide. The functions μ_i are continuous on M. Denote by $M^0 \subseteq M$ the set of points where the number of different eigenvalues of A is maximal on M. Since the functions μ_i are continuous, M^0 is open in M. Moreover, it was shown in [15] that M^0 is dense in M as well. The implicit function theorem now implies that μ_i are differentiable functions on M^0.

From Remark 3 and equation (7) we immediately obtain that g and $\bar g$ are affinely equivalent, if and only if all eigenvalues of A are constant. Suppose that g and $\bar g$ are not affinely equivalent, that is, there is a non-constant eigenvalue ρ of A with multiplicity $k \ge 1$. Let us choose a point $x_0 \in M^0$ such that $d\rho_{|x_0} \ne 0$, define $c := \rho(x_0)$ and consider the hypersurface $H = \{x \in U : \rho(x) = c\}$, where $U \subseteq M^0$ is a geodesically convex neighborhood of x_0. We think that U is sufficiently small such that $\mu(x) \ne c$ for all eigenvalues μ of A different from ρ and all $x \in U$.

Lemma 1. *There is a smooth nowhere vanishing $(0,2)$-tensor T on U such that on $U \setminus H$, T coincides with*

$$\operatorname{sgn}(\rho - c)|\det(A - cId)|^{\frac{1}{k}} g((A - cId)^{-1}.,.). \tag{10}$$

Proof. Let us denote by $\rho = \rho_1, \rho_2, ..., \rho_r$ the different eigenvalues of A on M^0 with multiplicities $k = k_1, k_2, ..., k_r$ respectively. Since the eigenspace distributions of A are differentiable on M^0, we can choose a local frame $\{U_1, ..., U_m\}$ on U, such that g and A are given by the matrices

$$g = \operatorname{diag}(1, ..., 1) \text{ and } A = \operatorname{diag}(\underbrace{\rho, ..., \rho}_{k \text{ times}}, ..., \underbrace{\rho_r, ..., \rho_r}_{k_r \text{ times}})$$

with respect to this frame. The tensor (10) can now be written as

$$\begin{aligned}
&\operatorname{sgn}(\rho - c)|\det(A - cId)|^{\frac{1}{k}} g(A - cId)^{-1} = \\
&= (\rho - c)\prod_{i=2}^{r} |\rho_i - c|^{\frac{k_i}{k}} \operatorname{diag}\Big(\underbrace{\frac{1}{\rho - c}, ..., \frac{1}{\rho - c}}_{k \text{ times}}, ..., \underbrace{\frac{1}{\rho_r - c}, ..., \frac{1}{\rho_r - c}}_{k_r \text{ times}}\Big) = \\
&= \prod_{i=2}^{r} |\rho_i - c|^{\frac{k_i}{k}} \operatorname{diag}\Big(\underbrace{1, ..., 1}_{k \text{ times}}, ..., \underbrace{\frac{\rho - c}{\rho_r - c}, ..., \frac{\rho - c}{\rho_r - c}}_{k_r \text{ times}}\Big).
\end{aligned} \tag{11}$$

Since $\rho_i \ne c$ on $U \subseteq M^0$ for $i = 2, ..., r$, we see that (11) is a smooth nowhere vanishing $(0,2)$-tensor on U. $\square$

Lemma 2. *The multiplicity of the non-constant eigenvalues of A is equal to $1 - \epsilon$.*

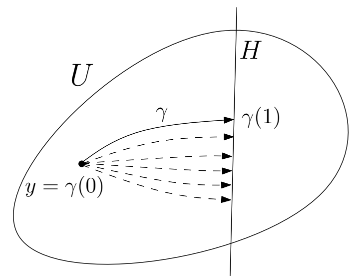

FIGURE 1. Case $\frac{1}{1-\epsilon} - \frac{1}{k} > 0$: We connect the point $y \in U \setminus H$ with the points in H by geodesics. The value of the integral F_c is zero on each of these geodesics.

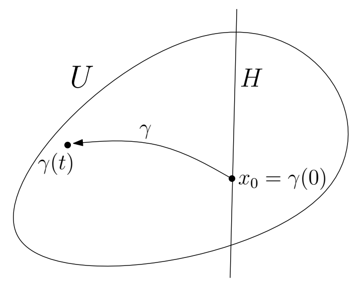

FIGURE 2. Case $\frac{1}{1-\epsilon} - \frac{1}{k} < 0$: For any geodesic γ starting in $x_0 \in H$ and leaving H, the value of the integral F_c along this geodesic is infinite.

Proof. Let us consider the integral $F_c : TM \to \mathbb{R}$ defined in equation (9). Using the tensor T from Lemma 1, we can write F_c as

$$F_c(X) = \underbrace{\operatorname{sgn}(\rho - c)|\det(A - cId)|^{\frac{1}{1-\epsilon}-\frac{1}{k}}}_{=:f_c} T(X, X), \quad X \in TM. \tag{12}$$

Our goal is to show that $\frac{1}{1-\epsilon} - \frac{1}{k} = 0$.

First suppose that $\frac{1}{1-\epsilon} - \frac{1}{k} > 0$ and let be $y \in U \setminus H$. We choose a geodesic $\gamma : [0, 1] \to U$ such that $y = \gamma(0)$ and $\gamma(1) \in H$, see figure 1. Since $\rho(\gamma(t)) \overset{t \to 1}{\longrightarrow} c$, we see from equation (12) that $f_c(\gamma(t)) \overset{t \to 1}{\longrightarrow} 0$. It follows that $F_c(\dot\gamma(t)) \overset{t \to 1}{\longrightarrow} 0$. On the other hand, since F_c is an integral for the geodesic flow of g (see Fact 2), the value $F_c(\dot\gamma(t))$ is independent of t and, hence, $F_c(\dot\gamma(0)) = 0$. We have shown that $F_c(\dot\gamma(0)) = 0$ for all initial velocities $\dot\gamma(0) \in T_yM$ of geodesics connecting y with points of H. Since H is a hypersurface, it follows that the quadric $\{X \in T_yM : F_c(X) = 0\}$ contains an open subset which implies that $F_c \equiv 0$ on T_yM. This is a contradiction to Lemma 1, since T is non-vanishing in y. We obtain that $\frac{1}{1-\epsilon} - \frac{1}{k} \le 0$.

Let us now treat the case when $\frac{1}{1-\epsilon} - \frac{1}{k} < 0$. We choose a vector $X \in T_{x_0}M$ which is not tangent to H and satisfies $T(X, X) \neq 0$. Such a vector exists, since $T_{x_0}M \setminus T_{x_0}H$ is open in $T_{x_0}M$ and T is not identically zero on $T_{x_0}M$ by Lemma 1. Let us consider the geodesic γ with $\gamma(0) = x_0$ and $\dot\gamma(0) = X$, see figure 2. Since $X \notin T_{x_0}H$, the geodesic γ has to leave H for $t > 0$. In a point $\gamma(t) \in U \setminus H$ the value $F_c(\dot\gamma(t))$ will be finite. On the other hand, since $f_c(\gamma(t)) \overset{t \to 0}{\longrightarrow} \infty$ and $T(\dot\gamma(0), \dot\gamma(0)) \neq 0$, we have $F_c(\dot\gamma(t)) \overset{t \to 0}{\longrightarrow} \infty$. Again this contradicts the fact that the value of F_c must remain constant along $\dot\gamma$ by Fact 2. We have shown that $\frac{1}{1-\epsilon} - \frac{1}{k} = 0$ and finally, Lemma 2 is proven. □

As a consequence of Lemma 2, if the metrics $g, \bar g$ are not affinely equivalent (i.e., at least one eigenvalue of A is non-constant), ϵ is an integer less or equal to zero. If $\epsilon \neq 0$, the condition $PQ = \epsilon Id$ in (1) implies that P is non-degenerate and by the first condition in (1), $g(P., .)$ is a non-degenerate 2-form on each eigenspace of A (note that A and P commute). This implies that for $\epsilon \neq 0$ the eigenspaces of A have even dimension, in particular, $1-\epsilon \in \{2,4,6,8,...\}$. Theorem 1 is proven.

2.3. **Proof of Theorem 2.** Let $g, \bar g$ be two PQ^ϵ-projective metrics and let A be the corresponding solution of equation (6) defined by (5). As it was already stated in the proof of Theorem 1, the eigenspace distributions of A are differentiable in a neighborhood of almost every point of M. First let us prove

Lemma 3. *Let X be an eigenvector of A corresponding to the eigenvalue ρ. If μ is another eigenvalue of A and $\rho \neq \mu$, then $X(\mu) = 0$. In particular,* $\operatorname{grad}\mu$ *is an eigenvector of A corresponding to the eigenvalue μ.*

Remark 6. Lemma 3 is known for projectively equivalent (Example 2) and h-projectively equivalent (Example 3) metrics. For projectively equivalent metrics it is a classical result which was already known to Levi-Civita [4]. For h-projectively equivalent metrics, it follows from [1, 8].

Proof. Let Y be an eigenvector field of A corresponding to the eigenvalue μ. For arbitrary $X \in TM$, we obtain $\nabla_X(AY) = \nabla_X(\mu Y) = X(\mu)Y + \mu\nabla_X Y$ and $\nabla_X(AY) = (\nabla_X A)Y + A\nabla_X Y$. Combining these equations and replacing the expression $(\nabla_X A)Y$ by (6) we obtain

$$(A-\mu Id)\nabla_X Y = X(\mu)Y - g(Y,X)\Lambda - g(Y,\Lambda)X - g(Y,QX)P\Lambda - g(Y,P\Lambda)QX. \tag{13}$$

Now let X be an eigenvector of A corresponding to the eigenvalue ρ and suppose that $\rho \neq \mu$. Since A is g-self-adjoint, the eigenspaces of A corresponding to different eigenvalues are orthogonal to each other. Moreover, since A and Q commute, Q leaves the eigenspaces of A invariant. Using (13) we obtain

$$(A-\mu Id)\nabla_X Y + g(Y,\Lambda)X + g(Y,P\Lambda)QX = X(\mu)Y.$$

Since the left-hand side is orthogonal to the μ-eigenspace of A, we necessarily have $X(\mu) = 0$. We have shown that $g(\operatorname{grad}\mu, X) = X(\mu) = 0$ for any eigenvalue μ and any eigenvector field X corresponding to an eigenvalue different form μ. This forces $\operatorname{grad}\mu$ to be contained in the eigenspace of A corresponding to μ. □

Now suppose that $\epsilon = 0$. Let us denote the non-constant eigenvalues of A by $\rho_1, ..., \rho_l$. Using Lemma 2, the corresponding eigenspaces are 1-dimensional and Lemma 3 implies that they are spanned by the gradients $\operatorname{grad}\rho_1, ..., \operatorname{grad}\rho_l$ respectively. Since P and A commute, P leaves the eigenspaces of A invariant, hence, $P\operatorname{grad}\rho_i = p_i \operatorname{grad}\rho_i$ for some real number p_i. Now P is skew with respect to g and we obtain $0 = g(\operatorname{grad}\rho_i, P\operatorname{grad}\rho_i) = p_i g(\operatorname{grad}\rho_i, \operatorname{grad}\rho_i)$ which implies that

$$P\operatorname{grad}\rho_i = 0.$$

On the other hand, by equation (7)

$$\Lambda = \frac{1}{2}\operatorname{grad}\operatorname{trace} A = \frac{1}{2}(\operatorname{grad}\rho_1 + ... + \operatorname{grad}\rho_l).$$

Combining the last two equations, we obtain $P\Lambda = 0$. It follows from Remark 4 that g and $\bar g$ are projectively equivalent and, hence, Theorem 2 is proven.

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