Polyhedral 3-manifolds of nonnegative curvature

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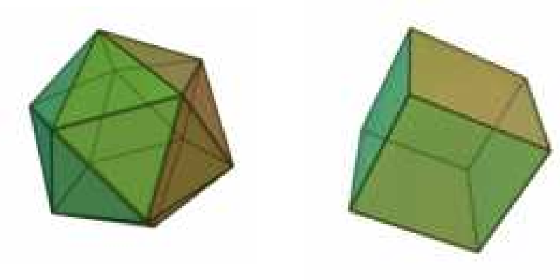
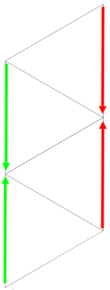
Definitions

Polyhedron = a convex hull of finitely many points in $\mathbb{R}^3$ with the induced metrics (= distance-function)





2- dim Analog

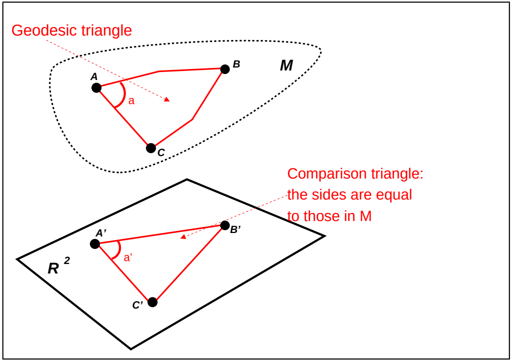
Polyhedral manifold =

- closed 3-manifold
- glued from polyhedra
- such that the glueing mappings are
  - face-to-face and
  - are isometries.

With the induced metric

Polyhedral manifold has nonnegative Alexandrov-curvature:

Geometric definition:



Def: $K \geq 0 \iff a \geq a'$
(for all geodesic triangles)

Combinatoric definition:

(Milka 1968)

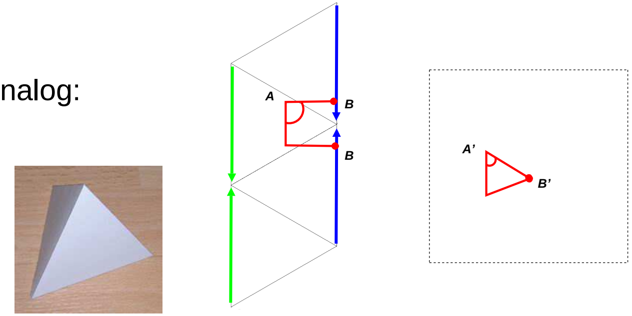
for every edge the sum of the dihedral angles around this edge is less than or equal to 2π.

(Since the polyhedra are glued face-to-face, the notion of edge is well defined)

Equivalence of definitions: Toponogov Globalisation Theorem:
(Burago-Gromov-Perelman 1992)

Example: Boundary of the convex (n+1)-polyhedron is a polyhedral n-manifold with $K \geq 0$.

2- dim Analog:



Alexandrov Theorem 1956: For dim 2, essentially all polyhedral 2-manifolds with $K \geq 0$ are (isometric to) boundaries of 3-polyhedra

Essentially all = modulo double cover, correct definition of the boundary of degenerate polyhedra, and forgetting about flat 2-manifolds.

In dim $n \geq 3$ one can construct polyhedral manifolds of $K \geq 0$ which are boundaries of no (n+1)-polyhedron.

The distance $d_g(x, y)$ on a Riemannian (M, g) is
$d_g(x, y) = \inf \left\{ L(\gamma) \ : \ \substack{\gamma \text{ is a smooth cur-} \\ \text{ve joining } x \text{ and } y} \right\},$
where $L(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$
for $\gamma : [a, b] \to M$.

Main theorem

(M^3, d) closed polyhedral, $K \geq 0$.

Then, $\forall \varepsilon > 0$ $\forall 0 \leq s < 1$

$\exists$ a smooth structure and a Riemannian metric g of nonnegatrive sectional curvature on M^3 such that

(i) (M^3, d_g) is Gromov-Hausdorff ε-close to (M^3, d).

$\Uparrow$ $\Uparrow$

(ii) (M^3, d_g) is ε-close to (M^3, d) in the C^0-sense.

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(iii) (M^3, d_g) is ε-close to (M^3, d) in the H^s-sense ($=$ $s-$Gromov-Hölder distance).

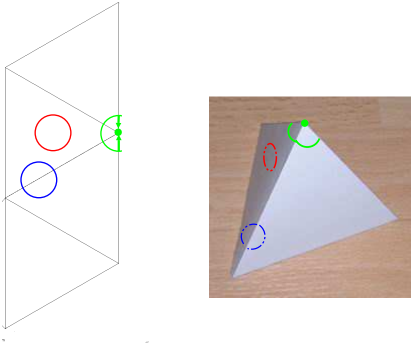
(ii) $\iff$ $\exists$ smooth Riemannian $(\tilde{M}^3, g)$ and a homeomorphism $\phi : M^3 \to \tilde{M}^3$ s.t. $\forall x, y \in M^3$ $|d(x, y) - d_g(\phi(x), \phi(y))| < \varepsilon$

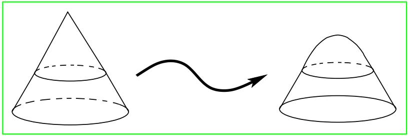
An edge will be called **essential**, if the sum of the dihedral angle around this edge is $< 2\pi$. A vertex will be called **essential**, if it is the endpoint of at least three essential edges.

(iv) In in addition there exists an essential vertex, then the metric g can be chosen to have positive sectional curvature.

Geometric Motivation:

(Alexandrov, Gauss) In dimension 2 the (analog of the) theorem is true: The metric in red and blue nighbourhoods is already Riemannian:





The metric of the green neighborhood is isometric to a neighborhood of the vertex in a cone and can be smoothed by replacing the cone (left) by a smooth "cap"

We see that we change the metric in a arbitrary small neighborhoods of the vertexes. Making the neighborhoods very small we make the Hausdorff-, C^0- and H^s- but not (Lipschitz-) distance between the initial and the Riemannian metrics arbitrary small.

Topological motivations

Polyhedral manifolds of $K \leq 0$ are useful for the inverstigation of hyperbolic manifolds (effective ways of constructions, minimal volumes, etc.) **(Thurston, S. Matveev, Freedman)**

Investigation of orbifolds **(Boileau, Leeb, Porti, Weiß)**

It is known that the existence of a Riemannian metric of positive section curvature is a strong topological obstruction.
Is the existence of a metric of nonnegative Alexandrov curvature also a strong obstruction? **(Alexandrov, Cheeger)**
(The question is natural: the property of metric to have bounded Alexandrov curvature survives under taking Hausdorff limit)

Corollary *Let (M, d) be a closed polyhedral 3-dimensional manifold of nonnegative curvature.*
Then, the manifold M can be finitely covered by S^3, $S^2 \times S^1$, or $S^1 \times S^1 \times S^1$.
Moreover, the manifold is homeomorphic to the quotient of one of the spaces S^3 or $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of fixed point free isometries in the standard metric.
Moreover, the existence of an essential vertex implies that M can be finitely covered by S^3.
Proof of Corollary: Main theorem $\Longrightarrow$ there exists a Riemannian metric of nonnegative curvature. Hamilton's Ricci-flow theorem $\Longrightarrow$ the manifold is as we claim.

Analytic motivation: geometric flows

How to define a natural flow (for example, Ricci flow) on a piece-wise Riemannian manifold (M, d)?
Suggestion (Gromov 1986):

- Consider a sequence g_n of Riemannian metrics approximating d: $\quad d_{g_n} \overset{n\to\infty}{\longrightarrow} d$
- Consider the Ricci-Flow-evolution g_n^t of the metrics g_n.
- Hope that
  - g_n^t converges to the a metric g^t, and
  - that the result is independent of the choice of approximations g_n

Works if one controls the lower bound of the curvature of g_n (Simon, 2002 – 2008).

Combinatoric motivation: triangulations of manifold

Luo Feng, Walkup, Lutz, Sullivan, Kühnel, Banchoff, Panov.

Possible question to study: *What is the simplest triangulation of a given manifold?*

Possible way to study: one can canonically supply every triangulated manifold with the structure of polyhedral manifold: let us think that all simplexes are regular. Then, the gluing mappings are automatically isometric. The simplest triangulation is one having the simplest metric.

Theorem (Lutz/Sullivan $\overset{\text{independently}}{---}$ Shevchishin/M$\sim$). *Let M be a triangulated closed 3-manifold. Assume for every edge of the triangulation the number of the simplices containing this edge is less than six. Then, M can be finitely covered by S^3.*

Proof of Lutz–Sullivan

They wrote a computer code describing all triangulations such that for every edge the number of the simplices containing this edge is less than six.

They (the computer) found that

- there are only finitely many such triangulations (precisely 4787),
- listed them all, and
- proved that in all cases the obtained manifold is covered be the sphere.

Our proof

The dihedral angle of every edge of the regular tetrahedron is

$$\arccos(1/3) \approx 70.53\frac{2\pi}{360}.$$

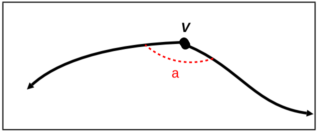
Since every edge lies in at most 5 simplices, for every edge the sum of dihedral angles around this edge is at most

$$5\times\arccos(1/3) \approx 352.66\frac{2\pi}{360} < 2\pi.$$

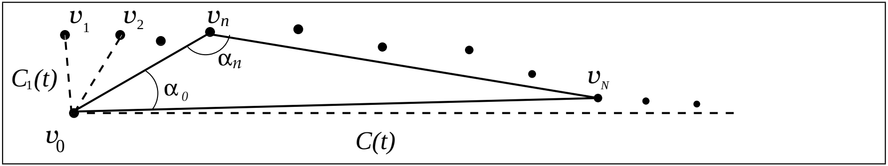
Thus, by our result, the manifold can be covered by S^3.

Folklor (Boileau, Leeb, Porti 2005): *If a polyhedral manifold of $K \geq 0$ has an essential vertex, then its fundamental group is finite.*

Metric characterisation of essential vertices v: $\exists \delta > 0$ s. t. for every two geodesics starting at v the angle between these geodesics $< \pi - \delta$.



Assume the fundamental group of M is infinite. Consider the universal cover $\widetilde{M}$ of M, the preimages $v_0, v_1, v_2, ... \in \widetilde{M}$ of v, and the geodesics c_i connecting v_0 and v_i. The sequence $c_i'(0)$ has a convergent subsequence.



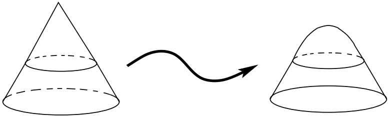
Take big numbers $n << N$ and consider the geodesic triangle $(v_0 v_n v_N)$ with v_n, v_N from the subsequence. The following statements are contradictive:

1. The angle α_0 is arbitrary small (because $c_n'(0) \xrightarrow{n\to\infty} 0$)
2. The angle α_n is less than $\pi - \delta$ (because the edge is essential)
3. For sufficiently big $N \gg n$, $\alpha_0 + \alpha_n$ can not be much smaller than π (because this is fulfilled in the comparison triangle)

Proof: Easy case: no essential vertices

Example: $M^3 \overset{\text{Isometric}}{=} \text{Surface\_of\_Cube} \times S^1$.

By Milka 1968 the essential edges are organized in circles, and the metric in a tubular neighborhood of the circles is isometric to the direct product of the metric of the cone and the circle. One can smooth this metric by replacing the vertex of the cone by a smooth cap:



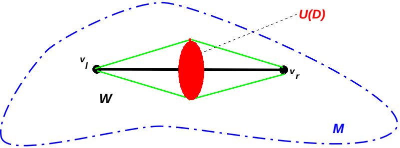
Warning: There exist polyhedral structures on S^3 with no essential vertices. On the other hand, by Corollary, if an essential vertex exists, then M is a finite quotient of S^3.

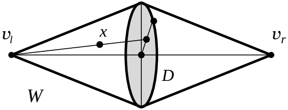
From now on: *Assume that an essential vertex exists.*

Proof: naive approach that does not work

In dim 2, one can smooth the metric in the neighborhood of vertices, and obtain a smooth metric of positive curvature. Can one mimic the proof in dim 3? Can one smooth the metric in a small tube neighborhood of a 1-sceleton?

The left (right) half of W is a flat cone over two-dimensional 1-cone.





Let us smooth the metrics in the right and in the left half by smoothing the conic singularities as here:  , and by glueing the metrics in a neighborhood of D by using partition of unity.

Good news: Metric is Riemannian outside of vertexes and has conic singularities near vertexes.

Bad news: There will necessary appear points such that the curvature is negative: Because in the quadrangle the angles at v_l, v_r are smaller than they were before smoothing, and the angles in other vertexes were not changed.

Very bad news: By Yau positive energy theorem, such problem will always appear: it is not possible to smooth the metric in the neighborhood of one edge only such that the manifold remains to be of nonnegative curvature.

Scheme of the proof

Step I: We show that (in the presence of at least one essential vertex) it is possible to slightly change the metric on the manifold in the class of polyhedral metrics such that

- all polyhedra are tetrahedra,
- all edges are essential.

Methods: Alexandrov Theorem + tricks

Step IIa: We replace every tetrahedron by the spherical tetrahedron of very small curvature $K = \frac{1}{R^2}$. The gluing functions remains isometric.

Step IIb: Then, we smooth the metric d_k near the edges as in the „naive approach". As result we obtain a metric of nonnegative curvature which is a Riemannian metric outside the vertexes and has k-conic singularities near the vertexes.

Methods: Analysis (Ansatz + work with inequalities)

Step III: We smooth the obtained metric near the vertexes with preserving the positivity of the curvature.

Methods: Alexandrov Theorem + convex geometry

Step I of the proof: *There exists a Lipschitz-small polyhedral deformation preserving the condition* $K \geq 0$ *such that after the deformation*

- all polyhedra are tetrahedra and
- all edges (and, therefore, all vertexes) are essential.

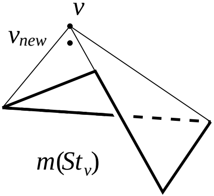
Remark. The claim is false in higher dimension:
Fact 1 (Cheeger, 1986): : If in a closed polyhedral manifold M of any dimension n all polyhedra are tetrahedra and all edges are essential, then M is a rational homology sphere with finite fundamental group.

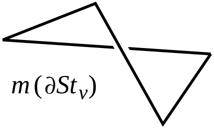
Fact 2. (Banchoff/Kühnel 1982) There exists a polyhedral metric d of non-negative Alexandrov curvature on $\mathbb{CP}^2$.

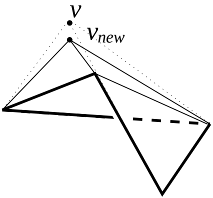
Conclusion: Step I is not always possible in dim 4.

Step I: main trick: „pressing by a finger“

We will change the metric and the decomposition into the polyhedra in a star of an essential vertex v. Take such vertex v. Consider its star St_v and the isometric embedding $m : St_v \to \mathbb{R}^4$ as the part of the boundary of a cone C_{old}.







Slightly move v inside the cone (v_{new}), and replace St_v by the boundary of the intersection of the convex hull of $C(v_{new}, m(\partial St_v))$ with C_{old}.

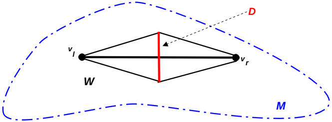
Theorem: *Applying this construction finitely many time we obtain all edges to be essential.*

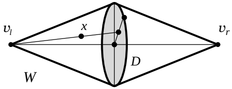
Step IIa of the proof: replacind tetrahedra by spherical tetrahedra

Fix very small $k = \frac{1}{R^2} > 0$. Replace each flat tetrahedron (P_j, d) by a spherical tetrahedron (P_j, d_k) of the curvature k with the same length of edges. Then (P_j, d_k) can be isometrically glued together yielding a **spherical polyhedral metric** d_k on M. For k small enough d_k still has only essential edges. By construction, the Alexandrov curvature $K \geq k$.

Step IIb of the proof: smoothing edges

As in the naive approach, consider the neighborhood W of every edge.





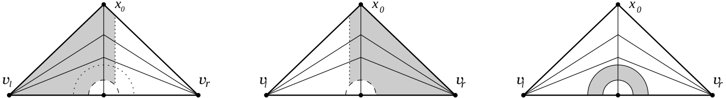
Theorem: There exists a Riemannian metric g with $K \geq 0$ inside W s.t.

- d_g coinsides with d_k near every $p \in \partial W \setminus \{v_r, v_l\}$
- d_g in a neighborhood of v_l, v_r is isometric to neighborhood of a vertex of a k-cone over a 2-sphere (S^2, g_{Riem}).

Normal proof of theorem: The metric in W is given by an explicit formula. It should be possible to write an explicit formula for g.

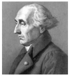
Our proof of theorem: We prove the existence: we act as in „the naive approach“: The left (right) half of W is a k-cone over two-dimensional 1-cone.

Let us smooth the metrics in the right and in the left half by smoothing the conic singularities as here: , and by glueing the metrics in a neighborhood of D by using this partition of unity.

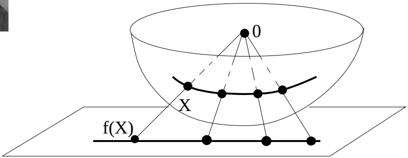


Step III: smoothing near edges

Petrunin's trick 2003: isometrically embed (Alexandrov theorem) a neighbourhood of a vertex (as a neighborhood of a vertex of a convex cone in S_k^4) and smooth as in 4-dim generalisation of  .



Additional observation (Lagrange 1789) that helps to have a usual-human-understandable proof: radial projection sends geodesics of the sphere to straight lines. In particular, it takes cones to cones and convex sets to convex sets.



Conclusions

We proved: polyhedral metrics of $K \geq 0$ can be approximated by Riemannian metrics of $K \geq 0$.

Which in particular allows to transwer all Hamilton's topological results: the 3-manifolds carrying polyhedral metrics of $K \geq 0$ are essentially quotients of the sphere.

Conjecture (Petrunin 2001/Kühnel 2004) *Every $n-$manifold with polyhedral metrics s.t. all polyhedra are tetrahedra and all sides of dimension $n-2$ are essential admits a Riemannian metric with positive curvature operator (and is therefore a space-form by Wilking, Böhm 2006).*

Thank you!!!