

ON PROJECTIVE EQUIVALENCE AND POINTWISE PROJECTIVE RELATION OF RANDERS METRICS.

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ABSTRACT. We correct a mistake in [19] and prove the natural generalization of the projective Lichnerowicz-Obata conjecture for Randers metrics.

1. INTRODUCTION

1.1. Definition and results. A Randers metric is a Finsler metric of the form

$$(1.1) \quad \begin{aligned} F(x, \xi) &= \sqrt{g(x)_{ij} \xi^i \xi^j} + \omega(x)_i \xi^i \\ &= \sqrt{g(\xi, \xi)} + \omega(\xi), \end{aligned}$$

where $g = g_{ij}$ is a Riemannian metric and $\omega = \omega_i$ is an 1-form. Here and everywhere in the paper we assume summation with respect to repeating indices. The assumption that F given by (1.1) is indeed a Finsler metric is equivalent to the condition that the g -norm of ω is less than one. Within the whole paper we assume that all objects we consider are at least C^2 -smooth,

By a *forward geodesic* of a Finsler metric F we understand a regular curve $x : I \rightarrow M$ such for any sufficiently close points $a, b \in I$, $a \leq b$ the restriction of the curve x to the interval $[a, b] \subseteq I$ is an extremal of the forward-length functional

$$(1.2) \quad L_F^+(c) := \int_a^b F(c(t), \dot{c}(t)) dt$$

in the set of all smooth curves $c : [a, b] \rightarrow M$ connecting $x(a)$ and $x(b)$. By a *backward geodesic* of a Finsler metric F we understand a regular curve $x : I \rightarrow M$ such for any sufficiently close points $a, b \in I$, $a \leq b$ the restriction of the curve x to the interval $[a, b] \subseteq I$ is an extremal of the backward-length functional

$$(1.3) \quad L_F^-(c) := \int_a^b F(c(t), -\dot{c}(t)) dt$$

in the set of all smooth curves $c : [a, b] \rightarrow M$ connecting $x(a)$ and $x(b)$.

Note that these definitions do not assume any preferred parameter on the geodesics: if $x(\tau)$ is a say forward geodesic, and $\tau(t)$ is a (orientation-preserving) reparameterisation with $\dot{\tau} := \frac{d\tau}{dt} > 0$, then $x(t) := x(\tau(t))$ is also

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a forward geodesic. As the examples show, the condition that the reparametrisation is orientation-preserving, i.e., the condition that $\dot{\tau} := \frac{d\tau}{dt} > 0$, is important though.

We will study the question when two Randers metrics F and \bar{F} are *projectively equivalent*, that is, when every forward-geodesic of F is a forward-geodesic of \bar{F} . Within our paper we will always assume that the dimension is at least two, since in dimension one all metrics are projectively equivalent.

Remark 1. Comparing (1.2) and (1.3) we see that for every forward geodesic $x(t)$, $t \in [-1, 1]$, the ‘reverse’ curve $\tilde{x}(t) := x(-t)$ is a backward-geodesic, and vice versa. Thus, if two metrics F and \bar{F} have the same forward-geodesics, they automatically have the same backward-geodesics.

Moreover, the forward-geodesics of any metric F are the backward-geodesics of the metric \tilde{F} given by $\tilde{F}(x, \xi) = F(x, -\xi)$ and vice versa. For the Randers metrics, the transformation $F \mapsto \tilde{F}$ reads $\sqrt{g(\xi, \xi) + \omega(\xi)} \mapsto \sqrt{g(\xi, \xi) - \omega(\xi)}$.

There exists the following ‘trivial’ examples of projectively equivalent Randers metrics:

Every Finsler metric F is projectively equivalent to the Finsler metric $\text{const} \cdot F$ for any constant $\text{const} > 0$. Indeed, forward and backward geodesics are extremals of (1.2), (1.3). Now, replacing F by $\text{const} \cdot F$ multiplies the length-functionals L^{pm} of all curves by const so extremals remain extremals. If the Finsler metric F is Randers, the operation $F \mapsto \text{const} \cdot F$ multiplies the metric g by const^2 and the form ω by const .

Any Finsler metric F is projectively equivalent to the Finsler metric $F + \sigma$, where σ is an arbitrary closed one-form such that $F(x, \xi) + \sigma(\xi) > 0$ for all tangent vectors $\xi \neq 0$. Indeed, a geodesic connecting two points x, y is an extremal of the length functionals $L^\pm(c)$ given by (1.2), (1.3), over all regular curves $c : [a, b] \rightarrow M$ with $c(a) = x$ and $c(b) = y$. Adding σ to the Finsler metric F changes the (forward and backward) length of all such curves in one homotopy class by adding a constant to it (the constant may depend on the homotopy class), which does not affect the property of a curve to be an extremal. If the Finsler metric F is Randers, the operation $F \mapsto F + \sigma$ does not change the metric g and adds the form σ to the form ω .

One can combine two examples above as follows: any Finsler metric F is projectively equivalent to the Finsler metric $\text{const} \cdot F + \sigma$, where σ is a closed 1-form and $\text{const} \in \mathbb{R}_{>0}$ such that $\text{const} \cdot F(x, \xi) + \sigma(\xi) > 0$ for all tangent vectors $\xi \neq 0$.

Now, if the forms ω and $\bar{\omega}$ are closed, the Randers metrics $F(x, \xi) = \sqrt{g(\xi, \xi) + \omega(\xi)}$ and $\bar{F}(x, \xi) = \sqrt{\bar{g}(\xi, \xi) + \bar{\omega}(\xi)}$ on one manifold M are projectively equivalent if and only if the Riemannian metrics g and \bar{g} are projectively equivalent. Indeed, as we explained above the geodesics of F are geodesics of the Riemannian metric g , the geodesics of $\sqrt{\bar{g}(\xi, \xi) + \bar{\omega}(\xi)}$ are the geodesics of the Riemannian metric \bar{g} , so that projective equivalence of F and \bar{F} is equivalent to the projective equivalence of g and \bar{g} .

Note that there are a lot of examples of projectively equivalent Riemannian metrics; the first examples were known already to Lagrange [7] and local classification of projectively equivalent metrics was known already to Levi-Civita [8].

One of the goals of this note is to show that the ‘trivial’ examples above give us all possibilities of projectively equivalent Randers metrics.

Theorem 1. *Let the Finsler metrics $\sqrt{g_{ij}\dot{x}^i\dot{x}^j} + \omega_i\dot{x}^i$ and $\sqrt{\bar{g}_{ij}\dot{x}^i\dot{x}^j} + \bar{\omega}_i\dot{x}^i$ on a connected manifold be projectively equivalent. Suppose at least one of the forms ω and $\bar{\omega}$ is not closed. Then, for a certain $\text{const} \in \mathbb{R}_{>0}$ we have $g = \text{const}^2 \cdot \bar{g}$ and the form $\omega - \text{const} \cdot \bar{\omega}$ is closed.*

Let us first remark that Theorem 1 follows from [5, Theorem 1.1]. The paper [5] deals with the magnetic systems and study the question when magnetic geodesics of one magnetic system (g, Ω) are reparameterized magnetic geodesics of another magnetic system $(\bar{g}, \bar{\Omega})$, see [5] for definitions. In particular, it was proved that if for positive numbers (energy levels) $E, \bar{E} \in \mathbb{R}$, every magnetic geodesic with energy E of one magnetic system, is, after a proper reparameterisation, a magnetic geodesic with energy \bar{E} of another magnetic system, then $\bar{g} = \text{const} \cdot g$ and $\bar{\Omega} = \overline{\text{const}} \cdot \Omega$ (the second constant $\overline{\text{const}}$ depends on const, E, \bar{E}), or $\Omega = \bar{\Omega} = 0$ and the metrics g and \bar{g} are projectively equivalent. Now, it is well known that forward geodesics of the Randers metric (1.1) are, after an appropriate orientation preserving reparameterisation, magnetic geodesics with energy $E = 1$ of the magnetic system $(g, \Omega = d\omega)$. In view of this, Theorem 1 is actually a corollary of [5, Theorem 1.1].

Theorem 1 is visually very close to [19, Theorem 2.4]: the only essential difference is that in the present paper we speak about *projective equivalence*, and the condition discussed in [19] is when two metrics are *pointwise projectively related* (see section 1.3 for definition). In the Riemannian case (or, more generally, if the case when Finsler metrics are reversible), these two conditions, projective equivalence and pointwise projective relation, coincide. For general Finsler metrics, and in particular for Randers metrics, projective equivalence and pointwise projective relation are different conditions and [19, Theorem 2.4] is wrong: we give a counterexample in section 1.3. We also discuss how one can modify Theorem 1 and [19, Theorem 2.4] such that they become correct, see Corollaries 4 and 5 from section 1.3.

Besides, the proof of [19, Theorem 2.4], even if one replaces pointwise projective relation projective equivalence, seems to have a certain mathematical gap, namely an important delicate and nontrivial step was not done (at least we did not find the place where it was discussed); our proof of Theorem 1 closes this gap. We comment on this in section 1.3.

As we mentioned above, we do not pretend that Theorem 1 is new since it is a direct corollary of [5, Theorem 1.1], though it seems to be unknown within Finsler geometries. New results of the papers are related to projective transformations. By *projective transformation* of (M, F) we understand a

diffeomorphism ϕ such that pullback of F is projectively equivalent to F . By *homothety* of (M, F) we understand a diffeomorphism ϕ such that pullback of F is proportional to F . Homotheties evidently send forward geodesics to forward geodesics and are therefore projective transformations. As a direct application of Theorem 1, we obtain

Corollary 1. If the form ω is not closed, every projective transformation of (1.1) on a connected manifold M is a homothety of the Riemannian metric g . In particular, if M is closed, every projective transformation of (1.1) is an isometry of the Riemannian metric g .

Many papers study the question when a Randers metric is projectively flat, i.e., when its forward geodesics are straight lines in a certain coordinate system. Combining Theorem 1 with the classical Beltrami Theorem (see e.g. [1, 9, 16]), we obtain the following wellknown statement

Corollary 2 (Folklore). The metric (1.1) is projectively flat if and only if g has constant sectional curvature and ω is closed.

In the case the manifold M is closed (= compact and without boundary), more can be said. We denote by $Proj(M, F)$ the group of the projective transformations of the Finsler manifold (M, F) and by $Proj_0(M, F)$ its connected component containing the identity.

Corollary 3. Let (M, F) be a closed connected Finsler manifold with F given by (1.1). Then, at least one of the following possibilities holds:

- (1) There exists a closed form $\hat{\lambda}$ such that $Proj_0(M, F)$ consists of isometries of the the Finsler metric $F(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi) - \hat{\lambda}(\xi)$, or
- (2) the form ω is closed and g has constant positive sectional curvature.

1.2. Motivation. One of our motivation was to correct mistakes in the paper [19]: to construct counterexamples and to formulate the correct statement. A part of this goal is to give a simple selfcontained proof of Theorem 1 which does not require Finsler machinery and therefore could be interesting for a bigger group of mathematicians.

Actually, a lot of papers discuss metrics that are pointwise projectively related, and many of them have the same mistake as [19]: in the proofs, the authors actually use that metrics are projectively equivalent, but formulate the results assuming the metrics are pointwise projectively related. If for every forward geodesic $x : [-1, 1] \rightarrow M$ the reverse curve $x(-t)$ is also a forward geodesic, (for example, when the metrics are reversible), then the results remain correct; but in the general case many papers on pointwise projectively related metrics are wrong and in many cases in the papers it is not even mentioned whether the authors speak about all metrics or restrict themselves to the case when the metrics are reversible. Since the Randers metrics are nonreversible, this typical mistake is clearly seen in the case of Randers metrics and we have chosen [19] to demonstrate it.

Besides, we think that Corollary 3 is deserved to be publishes, since it is a natural generalisation of the classical projective Lichnerowicz-Obata conjectures for Randers metrics, see [15, 18, 6, 17] where the Riemannian version of the conjecture was formulated and proved under certain additional geometric assumptions, and [10, 11, 12] where the Riemannian version of the conjecture was proved in the full generality.

Additional motivation to study projective equivalence and projective transformations came from mathematical relativity and lorentz differential geometry: it was observed that the light-line geodesics of a stationary, standard spacetime can be described with the help of Randers metrics on a manifold of dimension one less. This observation is called the Stationary-Randers-Correspondence and it is nowadays a hot topic in the Lorentz differential geometry since one can effectively apply it, see for example [2, 3, 4, 13]. The projective transformations of the Randers metrics correspond then to the conformal transformations of the initial Lorentz metric preserving the integral curves of the Killing vector field, so one can directly apply our results.

1.3. Projective equivalence versus pointwise projective relation. By [19], two Finsler metrics F and \bar{F} on one manifold M are *pointwise projectively related*, if they have the same geodesics as point sets. The difference between this definition and our definition of projective equivalence is that in our definition we also require that the orientation of the forward geodesics is the same in both metrics. In particular, the metrics F and \bar{F} from Remark 1, such that every forward geodesic of the first is a backward geodesic of the second, are pointwise projectively related according to the definition from [19]. This allows one to construct immediately a counterexample to [19, Theorem 2.4].

Example 1. Take any Riemannian metric g and any form ω such that it is not closed. Next, consider the Randers metrics $F(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi)$ and $\bar{F}(x, \xi) = \sqrt{g(\xi, \xi)} - \omega(\xi)$ are pointwise projectively related, since for every forward geodesic of F is a backward geodesic of \bar{F} and every backward geodesic of F is a forward geodesic of \bar{F} . Since the Riemannian parts of these Finsler metrics coincide, [19, Theorem 2.4] claims that the form $\omega - (-\omega) = 2\omega$ is closed which is not the case.

The following two corollaries are an attempt to correct the statement of [19, Theorem 2.4]

Corollary 4. Suppose two Randers metrics $F(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi)$ and $\bar{F}(x, \xi) = \sqrt{\bar{g}(\xi, \xi)} + \bar{\omega}(\xi)$ on one connected manifold M are pointwise projectively related. Assume in addition that the set M^0 of the points of M such that the differential $d\omega$ is not zero is connected. Then, there exists a positive const $\in \mathbb{R}$ such that at least one of following statements holds at all points of the manifold:

- (1) $\bar{g} = \text{const}^2 \cdot g$ and $\bar{\omega} - \text{const} \cdot \omega$ is a closed form, or

(2) $\bar{g} = \text{const}^2 \cdot g$ and $\bar{\omega} + \text{const} \cdot \omega$ is a closed form.

It is important though that the set M^0 is connected: Indeed, as the following example shows, the cases (1), (2) of Theorem 4 could hold simultaneously in different regions of one manifold.

Example 2. Consider the ray $S := \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ and } y \leq 2\}$. Take $M = \mathbb{R}^2 \setminus S$ with the flat metric $g = dx^2 + dy^2$. Consider two balls B_+ and B_- around the points $(+1, 0)$ and $(-1, 0)$ of radius $\frac{1}{2}$. Next, take a smooth form ω on M such that ω vanishes on $M \setminus (B_+ \cup B_-)$, such that there exists at least one point of every ball such that $d\omega \neq 0$, and such that the g -norm of ω is less than 1 at every point. Next, define the form $\bar{\omega}$ as follows: set

$$\begin{cases} \bar{\omega} = -\omega & \text{at } p \in B_- \\ \bar{\omega} = \omega & \text{at } p \notin B_- \end{cases}$$

The form $\bar{\omega}$ is evidently smooth; the metrics $F(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi)$ and $\bar{F}(x, \xi) = \sqrt{\bar{g}(\xi, \xi)} + \bar{\omega}(\xi)$ are pointwise projectively related though none of the of the cases listed in Corollary 4 holds on the whole manifold.

Corollary 5. Suppose two Randers metrics $F(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi)$ and $\bar{F}(x, \xi) = \sqrt{\bar{g}(\xi, \xi)} + \bar{\omega}(\xi)$ on a connected manifold are pointwise projectively related. Suppose the form ω is not closed. Then, there exists a positive constant $\text{const} \in \mathbb{R}$ such that $g = \text{const}^2 \cdot \bar{g}$ and such that for every point $x \in M$ we have $d\omega = \text{const} \cdot d\bar{\omega}$ or $d\omega = -\text{const} \cdot d\bar{\omega}$.

Let us now explain, as announced in the introduction, one more mathematical difficulty with the proof of [19, Theorem 2.4]. Authors proved (in our notation and assuming that they actually work with projectively equivalent metrics)

- that at the points where the metrics g and \bar{g} are proportional with possibly a nonconstant coefficient, the coefficient of proportionality is actually a constant, $g = \text{const}^2 \cdot \bar{g}$, and the form $\omega - \text{const} \cdot \bar{\omega}$ is closed and
- that at the points such that the forms ω and $\bar{\omega}$ are closed the metric g and \bar{g} are projectively equivalent.

These two observations do not immediately imply that one of these two conditions holds on the whole manifold or even, if they work locally, at all points of a sufficiently small neighborhood of arbitrary point. Indeed, we could conceive of a two Randers metrics $\sqrt{g(\xi, \xi)} + \omega(\xi)$ and $\sqrt{\bar{g}(\xi, \xi)} + \bar{\omega}(\xi)$ such that at certain points of the manifold the metrics g and \bar{g} are projectively equivalent but nonproportional (the set of such points is open), and at certain points the metrics g and \bar{g} are proportional (the set of such points is evidently close).

We overcome this difficulty by using the (nontrivial) result of [14, Corollary 2], which implies that if two projectively equivalent metrics are proportional on a certain open set then they are proportional on the whole manifold

(assumed connected). It is not the only possibility to overcome this difficulty, but at the present point we do not know any of them which is completely trivial; since this difficulty is not addressed in the proof of [19, Theorem 2.4] we suppose that the authors simply overseen the difficulty; our paper closes this gap.

2. PROOFS.

2.1. Proof of Theorem 1. Recall that every forward geodesic $x(t)$ of a metrics is an extremal of the length functional $L(c) = \int_a^b F(c(t), \dot{c}(t))dt$, and, therefore, is a solutions of the Euler-Lagrange equation

$$(2.1) \quad \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} - \frac{\partial F}{\partial x} = 0.$$

For the Randers metric (1.1), the equation (2.1) reads

$$(2.2) \quad g_{ip} \ddot{x}^p \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) + g_{ip} \dot{x}^p \frac{d}{dt} \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) + \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) \frac{\partial g_{ip}}{\partial x^k} \dot{x}^k \dot{x}^p + \frac{\partial \omega_i}{\partial x^k} \dot{x}^k - \frac{\partial \omega_k}{\partial x^i} \dot{x}^k - \frac{1}{2\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \dot{x}^p \dot{x}^q \frac{\partial g_{pq}}{\partial x^i} = 0.$$

Multiplying this equation by g^{ij} (the inverse matrix to g_{ij}), we obtain

$$(2.3) \quad \ddot{x}^j \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) + \dot{x}^j \frac{d}{dt} \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) + g^{ij} \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) \frac{\partial g_{ip}}{\partial x^k} \dot{x}^k \dot{x}^p + g^{ij} \left(\frac{\partial \omega_i}{\partial x^k} - \frac{\partial \omega_k}{\partial x^i} \right) \dot{x}^k - g^{ij} \frac{1}{2\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \dot{x}^p \dot{x}^q \frac{\partial g_{pq}}{\partial x^i} = 0.$$

It is easy to check by calculations and is evident geometrically that for every solution $x(\tau)$ of (2.2) and for every time-reparameterization $\tau(t)$ with $\dot{\tau} > 0$ the curve $x(\tau(t))$ is also a forward geodesic. Thus, if $\sqrt{g_{ij} \dot{x}^i \dot{x}^j} + \omega_i \dot{x}^i$ and $\sqrt{\bar{g}_{ij} \dot{x}^i \dot{x}^j} + \bar{\omega}_i \dot{x}^i$ are projectively equivalent, then every solution $x(t)$ of (2.2) also satisfies

$$(2.4) \quad \ddot{x}^j \left(\frac{1}{\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m}} \right) + \dot{x}^j \frac{d}{dt} \left(\frac{1}{\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m}} \right) + \bar{g}^{ij} \left(\frac{1}{\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m}} \right) \frac{\partial \bar{g}_{ip}}{\partial x^k} \dot{x}^k \dot{x}^p + \bar{g}^{ij} \left(\frac{\partial \bar{\omega}_i}{\partial x^k} - \frac{\partial \bar{\omega}_k}{\partial x^i} \right) \dot{x}^k - \bar{g}^{ij} \frac{1}{2\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m}} \dot{x}^p \dot{x}^q \frac{\partial \bar{g}_{pq}}{\partial x^i} = 0,$$

where \bar{g}^{ij} is the inverse matrix to \bar{g}_{ij} . We now multiply the equation (2.4) by $\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m}$ and subtract the equation (2.3) multiplied by $\sqrt{g_{km} \dot{x}^k \dot{x}^m}$ to obtain

$$(2.5) \quad \begin{aligned} & \dot{x}^j \left(\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m} \frac{d}{dt} \left(\frac{1}{\sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m}} \right) - \sqrt{g_{km} \dot{x}^k \dot{x}^m} \frac{d}{dt} \left(\frac{1}{\sqrt{g_{km} \dot{x}^k \dot{x}^m}} \right) \right) \\ & + \dot{x}^k \left(\bar{g}^{ij} \sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m} \left(\frac{\partial \bar{\omega}_i}{\partial x^k} - \frac{\partial \bar{\omega}_k}{\partial x^i} \right) - g^{ij} \sqrt{g_{km} \dot{x}^k \dot{x}^m} \left(\frac{\partial \omega_i}{\partial x^k} - \frac{\partial \omega_k}{\partial x^i} \right) \right) \\ & + \dot{x}^k \dot{x}^p \left(\bar{g}^{ij} \frac{\partial \bar{g}_{ip}}{\partial x^k} - g^{ij} \frac{\partial g_{ip}}{\partial x^k} \right) - \dot{x}^p \dot{x}^q \left(\bar{g}^{ij} \frac{1}{2} \frac{\partial \bar{g}_{pq}}{\partial x^i} + g^{ij} \frac{1}{2} \frac{\partial g_{pq}}{\partial x^i} \right) = 0. \end{aligned}$$

In the equation above, the following logic in rearranging the terms was used: consider a forward-geodesic \tilde{x} such that $\tilde{x}(0) = x(0)$ and such that $\dot{\tilde{x}}(0) = -\dot{x}(0)$. Then, at $t = 0$, the first lines of the equation (2.5) and its analog for \tilde{x} are proportional to $\dot{x}(0) = -\dot{\tilde{x}}(0)$. The second line of the equation (2.5) is minus its analog for \tilde{x} . The remaining third line of the equation (2.5) coincides with its analog for \tilde{x} .

Then, subtracting the equation (2.5) from its analog for \tilde{x} at $t = 0$ we obtain (at $t = 0$)

$$-\dot{x}^j f(x(0), \dot{x}(0)) + \dot{x}^k \left(\bar{g}^{ij} \sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m} \left(\frac{\partial \bar{\omega}_i}{\partial x^k} - \frac{\partial \bar{\omega}_k}{\partial x^i} \right) - g^{ij} \sqrt{g_{km} \dot{x}^k \dot{x}^m} \left(\frac{\partial \omega_i}{\partial x^k} - \frac{\partial \omega_k}{\partial x^i} \right) \right) = 0,$$

where

$$\begin{aligned} f(x(0), \dot{x}(0)) = & \sqrt{\bar{g}_{km} \dot{x}^k \dot{x}^m} \left(\frac{d}{dt} \Big|_{t=0} \left(\frac{1}{\sqrt{\bar{g}(x)_{km} \dot{x}^k \dot{x}^m}} \right) - \frac{d}{dt} \Big|_{t=0} \left(\frac{1}{\sqrt{\bar{g}(\tilde{x})_{km} \dot{\tilde{x}}^k \dot{\tilde{x}}^m}} \right) \right) \\ & - \sqrt{g_{km} \dot{x}^k \dot{x}^m} \left(\frac{d}{dt} \Big|_{t=0} \left(\frac{1}{\sqrt{g(x)_{km} \dot{x}^k \dot{x}^m}} \right) - \frac{d}{dt} \Big|_{t=0} \left(\frac{1}{\sqrt{g(\tilde{x})_{km} \dot{\tilde{x}}^k \dot{\tilde{x}}^m}} \right) \right) \end{aligned}$$

Since this equation is fulfilled for every geodesic, for every tangent vector v we have

$$(2.6) \quad v^j f(v) = \sqrt{\bar{K}(v)} \bar{g}^{ij} \bar{L}_{ik} v^k - \sqrt{K(v)} g^{ij} L_{ik} v^k.$$

In the above equation,

$$K(v) = g_{pq} v^p v^q, \quad \bar{K}(v) = \bar{g}_{pq} v^p v^q, \quad L_{ik} = d\omega = \frac{\partial \omega_i}{\partial x^k} - \frac{\partial \omega_k}{\partial x^i}, \quad \bar{L}_{ik} = d\bar{\omega} = \frac{\partial \bar{\omega}_i}{\partial x^k} - \frac{\partial \bar{\omega}_k}{\partial x^i}.$$

Note that the matrices L_{ij} and \bar{L}_{ij} are scew-symmetric.

Let us view this equation as a system of algebraic equations in a certain tangent space \mathbb{R}^n ; we will show that for every symmetric positive definite matrices g_{ij} and \bar{g}_{ij} and for every scew-symmetric matrices L, \bar{L} there exist the following possibilities only:

- The matrix g_{ij} is proportional to the matrix \bar{g}_{ij} and the matrix L_{ij} is proportional to the matrix \bar{L}_{ij} with the same coefficient of proportionality, or
- The matrices L_{ij} and \bar{L}_{ij} are zero.

Indeed, we multiply the equation (2.6) by $v^p g_{jp}$, and using that $v^p g_{jp} \bar{g}^{ij} \bar{L}_{ik} v^k = v^i v^k L_{ik} = 0$ because of scw-symmetry of L , we obtain

$$K(v)f(v) = -\sqrt{\bar{K}(v)} g_{jp} \bar{g}^{ij} \bar{L}_{ik} v^k v^p.$$

Similarly, multiplying the equation (2.6) by $v^p \bar{g}_{jp}$, we obtain

$$\bar{K}(v)f(v) = \sqrt{K(v)} v^p \bar{g}_{jp} g^{ij} L_{ik} v^k.$$

Combining the last two equations we obtain

$$(2.7) \quad \bar{K}(v)^3 (g_{jp} \bar{g}^{ij} \bar{L}_{ik} v^k v^p)^2 = K(v)^3 (\bar{g}_{jp} g^{ij} L_{ik} v^k v^p)^2.$$

Since the algebraic expression $K(v)$ is irreducible (over \mathbb{R}), if $g_{jp} \bar{g}^{ij} \bar{L}_{ik} v^k v^p \neq 0$, the equation (2.7) implies that $K(v) = \alpha \bar{K}(v)$ implying $\bar{g} = \alpha^2 \cdot g$ for a certain $\alpha > 0$. In this case, the equation (2.6) implies $\bar{L}_{ij} = \alpha L_{ij}$.

Thus, at every point p of our manifold, we have one of the two above possibilities. If at all points the second possibility takes places, the 1-forms ω and $\bar{\omega}$ are closed and the Riemannian metrics g and \bar{g} are geodesically equivalent. If at least at one point the first possibility takes place, then in every point of a small neighborhood of the point the first possibility takes place, so that the differential of the forms $d\omega$ and $d\bar{\omega}$ are proportional at every point of a small neighborhood, $d\omega = \alpha(x)d\bar{\omega}$. The function α is then a constant implying the Finsler metrics are homothetic.

Then, the metrics g and \bar{g} are projectively equivalent: indeed, in the neighborhood consisting of the points where the 1-forms ω and $\bar{\omega}$ are closed they are projectively equivalent as we explained above, and at the neighborhoods where the differential of at least one of the forms $\omega, \bar{\omega}$ is not closed they are even proportional. Now, by [14, Corollary 2], if the metrics are projectively equivalent everywhere and nonproportional in some neighborhood, their restrictions to every neighborhood are nonproportional. Thus, in this case $d\omega = d\bar{\omega} = 0$ at every point of the manifold. Theorem 1 is proved.

2.2. Proof of Corollaries 1,2,3. Corollary 1 follows immediately from Theorem 1: if $\phi : M \rightarrow M$ is a local projective transformation of the metric (1.1) on a connected manifold M , then, by the definition of projective transformations, the pullback $\phi^*(F)$ is projectively equivalent to F . Then, by Theorem 1, if the form ω is not closed, we have $\phi^*(g) = \text{const} \cdot g$ implying ϕ is a homothety for g . Corollary 1 is proved.

In order to prove Corollary 2, we will use that the straight lines are geodesics of the standard flat Riemannian metric which we denote by g^{flat} . Then, a projectively flat metric is projectively equivalent to the Randers metric $F^{flat}(x, \xi) := \sqrt{g^{flat}(\xi, \xi)}$. Then, by Theorem 1, the form ω is closed, and the Riemannian metric g is projectively flat. By the classical Beltrami Theorem (see e.g. [1, 9, 16]), the metric g has constant sectional curvature. Corollary 2 is proved.

Let us now prove Corollary 3. We assume that (M, F) is a closed connected Finsler manifold with F given by (1.1) and denote by $Proj(M, F)$ its group

of projective transformations and by $Proj_0(M, F)$ the connected component of this group containing the identity.

Suppose first the form ω is closed. Then, every projective transformation of the Randers metric (1.1) is a projective transformation of the Riemannian metric g . Then, if $Proj_0(M, F)$ contains not only isometries, the metric g has constant positive sectional curvature by the Riemannian projective Obata conjecture (proven in [12, Corollary 1], [11, Theorem 1] and [10, Theorem 1]) and we are done.

Assume now the form ω is not closed. Then, by Theorem (1), every element of $Proj_0(M, F)$ is an isometry of g which in particular implies that the group $Proj_0(M, F)$ is compact. We consider an invariant measure $d\mu = d\mu(\phi)$ on $Proj_0(M, F)$ normalized such that

$$(2.8) \quad \int_{\phi \in Proj_0(M, F)} d\mu(\phi) = 1.$$

Consider the 1-form $\hat{\omega}$ given by the formula

$$\hat{\omega}(\xi) = \int_{\phi \in Proj_0(M, F)} \phi^* \omega(\xi) d\mu(\phi).$$

Here $\phi^* \omega$ denotes the pullback of the form ω with respect to the isometry $\phi \in Proj_0(M, F)$.

Let us show that the form $\hat{\omega} - \omega$ is closed. In view of (2.8), we have

$$(2.9) \quad \omega(\xi) - \hat{\omega}(\xi) = \int_{\phi \in Proj_0(M, F)} (\omega - \phi^* \omega)(\xi) d\mu(\phi).$$

Since for every $\phi \in Proj_0(M, F)$ the form $\lambda^\phi := \omega - \phi^* \omega$ is closed by Theorem 1, the form $\hat{\lambda} := \omega - \hat{\omega}$ is also closed. Indeed, let λ_i^ϕ be the components of the form $\lambda^\phi := \omega - \phi^* \omega$ in a local coordinate system (x^1, \dots, x^n) . Then, the components $\hat{\lambda}_i$ of $\omega - \hat{\omega}$ are given by the formula

$$\hat{\lambda}_i = \int_{\phi \in Proj_0(M, F)} \lambda_i^\phi d\mu(\phi).$$

Differentiating this equation w.r.t. x^j , we obtain

$$\frac{\partial}{\partial x^j} \hat{\lambda}_i = \int_{\phi \in Proj_0(M, F)} \frac{\partial}{\partial x^j} \lambda_i^\phi d\mu(\phi).$$

Now, since for every $\phi \in Proj_0(M, F)$, the form λ^ϕ is closed, we have $\frac{\partial}{\partial x^j} \lambda_i^\phi = \frac{\partial}{\partial x^i} \lambda_j^\phi$ implying $\frac{\partial}{\partial x^j} \hat{\lambda}_i = \frac{\partial}{\partial x^i} \hat{\lambda}_j$ implying $\hat{\lambda}$ is a closed form.

By construction, the form $\hat{\lambda}$ satisfies the property that $\omega - \hat{\lambda}$ is invariant with respect to $Proj_0(M, F)$, so the group $Proj_0(M, F)$ consists of isometries of the Finsler metric $\hat{F}(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi) - \hat{\lambda}(\xi)$. Corollary 3 is proved.

2.3. Proof of Corollaries 4, 5. We will prove the Corollaries 4, 5 simultaneously. Within this section we assume that $F(x, \xi) = \sqrt{g(\xi, \xi)} + \omega(\xi)$ and $\bar{F}(x, \xi) = \sqrt{\bar{g}(\xi, \xi)} + \bar{\omega}(\xi)$ are pointwise projectively related metrics on a connected manifold M and we denote by M^0 (resp. \bar{M}^0) the set of the points of M where the differential of ω (resp. of $\bar{\omega}$) does not vanish. M^0 and \bar{M}^0 are obviously open.

Let us first observe that $M^0 = \bar{M}^0$. In order to do it, consider $p \in M^0$ and a vector $\xi \in T_p M$ such that $d\omega(\xi, \cdot)$ (viewed as an 1-form) does not vanish. Then, the forward and backward geodesic segments $x(t)$ and $\tilde{x}(t)$ such that $x(0) = \tilde{x}(0) = p$ and $\dot{x}(0) = \dot{\tilde{x}}(0) = \xi$ are two different (even after a reparameterisation) curves. The curves are tangent at the point $x = 0$.

Indeed, in the proof of Theorem 1 we found an ODE (2.3) for forward geodesics. Analogically, one can find an ODE for backward geodesics: it is similar to (2.3), the only difference is that the term $g^{ij} \left(\frac{\partial \omega_j}{\partial x^k} - \frac{\partial \omega_k}{\partial x^j} \right) \dot{x}^k$ comes with the minus sign. We see that the difference between $\ddot{x}(0)$ and $\ddot{\tilde{x}}(0)$ is (in a local coordinate system) a vector which is not proportional to $\dot{x}(0) = \dot{\tilde{x}}(0) = \xi$. Then, in a local coordinate system, the geodesic segments have different curvatures¹ at the point $x(0)$. Then, the curves $x(t)$ and $\tilde{x}(\tau)$ do not have intersections for small $t \neq 0, \tau \neq 0$. Moreover, would $d\omega(\xi, \cdot) = 0$, the second derivatives $\ddot{x}(0)$ and $\ddot{\tilde{x}}(0)$ would coincide implying the curvatures of $x(t)$ and $\tilde{x}(t)$ coincide at $t = 0$.

Now consider the geodesics of \bar{F} . There must be one (forward or backward) geodesic of \bar{F} that, after an orientation preserving reparameterisation, coincides with $x(t)$, an one geodesic of \bar{F} that, after an orientation preserving reparameterisation, coincides with $\tilde{x}(t)$. Since the curvatures of these two geodesics are different at the point p , $d\bar{\omega}(\xi, \cdot) \neq 0$ at the point p . Finally, an arbitrary point $p \in M^0$ also lies in \bar{M}^0 , so that $M^0 \subseteq \bar{M}^0$. Similarly one proves $M^0 \supseteq \bar{M}^0$, so M^0 and \bar{M}^0 coincide.

Let us now show that, on M^0 , $d\omega = \text{const} \cdot d\bar{\omega}$. In fact, we explain how one can modify the proof of Theorem 1 to obtain this result. As we explained above, the forward and backward geodesic segments $x(t)$ and $\tilde{x}(t)$ such that $x(0) = \tilde{x}(0)$ and $\dot{x}(0) = \dot{\tilde{x}}(0)$ with $d\omega(\dot{x}(0), \cdot) \neq 0$ are geometrically different for small $t \neq 0$; then one of them is a forward geodesic of \bar{F} and another is a backward geodesic of \bar{F} . We call $\xi \in T_p M, p \in M^0$ a p -positive point, if the forward geodesic $x(t)$ with $x(0) = p$ and $\dot{x}(0) = \xi$ is a forward geodesic of \bar{F} , and p -negative otherwise. Next, we consider two subsets of $T_p M$:

$$S_+ := \{\xi \in T_p M \mid d\omega(\xi, \cdot) \neq 0; d\bar{\omega}(\xi, \cdot) \neq 0; \xi \text{ is } p\text{-positive}\},$$

$$S_- := \{\xi \in T_p M \mid d\omega(\xi, \cdot) \neq 0; d\bar{\omega}(\xi, \cdot) \neq 0; \xi \text{ is } p\text{-negative}\}.$$

¹In order to define a curvature, we need to fix an euclidean structure in the neighborhood of p . The curvature depends on the euclidean structure, but the property of curvatures (considered as vectors orthogonal to ξ) to be different does not depend on the choice of the euclidean structure

The closure of one of these two subsets contains a nonempty open subset $U \subset T_p M$; let us first assume that the closure of S_+ contains a nonempty open subset of $T_p M$. Then, as in the proof of Theorem 1, we obtain that (2.7) is valid for every $v \in S_+$ (here we use a trivial fact that if $v \in S_+$ then $-v \in S_+$). Since (2.7) is an algebraic condition, it must be then valid for all $v \in T_p M$. Then, arguing as in the proof of Theorem 1, we conclude that at the point p we have $\bar{g} = \alpha^2 \cdot g$ and $d\bar{\omega} = \alpha \cdot d\omega$ for some positive α . Now, if the closure of S_- contains a nonempty open subset of $T_p M$, we obtain $\bar{g} = \alpha^2 \cdot g$ and $d\bar{\omega} = \alpha \cdot d\omega$ for some negative α . At M^0 , α must be a smooth nonvanishing function as the coefficient of the proportionality of two nonvanishing tensors; then the sign of α is the same of at all points of every connected component of M^0 . Since the forms $d\bar{\omega}$ and $d\omega$ are closed, the coefficient of the proportionality, i.e., α is a constant.

This implies that the Riemannian metrics g and \bar{g} are projectively equivalent, since at the points of M^0 they are even proportional, and at the inner points of $M \setminus M^0$ the geodesics of g are geodesics of F and geodesics of \bar{g} are geodesics of \bar{F} . By [14, Corollary 2], the Riemannian metrics g and \bar{g} are proportional at all points of the manifold so that $\bar{g} = \text{const}^2 \cdot g$ on the whole manifold, as we claim in Corollaries 4, 5. Comparing this with the condition $\bar{g} = \alpha^2 \cdot g$ and $d\bar{\omega} = \alpha \cdot d\omega$ proved above for all points of each connected component of M^0 , we obtain that at every connected component of M^0 we have $d\bar{\omega} = +\text{const} \cdot d\omega$ or $d\bar{\omega} = -\text{const} \cdot d\omega$ (the sign can be different in different connected components of M^0 as Example 2 shows) as we claim in Corollaries 4, 5. At the points of $M \setminus M^0$ we have $d\bar{\omega} = d\omega = 0$ so both conditions $d\bar{\omega} = +\text{const} \cdot d\omega$ and $d\bar{\omega} = -\text{const} \cdot d\omega$ are fulfilled. Corollaries 4, 5 are proved.

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