

DIFFERENTIAL INVARIANTS FOR CUBIC INTEGRALS OF GEODESIC FLOWS ON SURFACES

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ABSTRACT. We construct differential invariants that vanish if and only if the geodesic flow of a 2-dimensional metric admits an integral of 3rd degree in momenta with a given Birkhoff-Kolokoltsov 3-codifferential.

1. INTRODUCTION

1.1. Definitions and results. Let S be a surface (i.e., 2-dimensional real manifold) equipped with a Riemannian metric g given in local coordinates by $g = \sum_{i,j} g_{ij} dx^i dx^j$. Since the metric g allows us to identify the tangent and cotangent bundles of S , we have a scalar product and a norm on every cotangent plane. The *geodesic flow* of the metric g is the Hamiltonian system with the Hamiltonian $H := \frac{1}{2} |\vec{p}|^2 = \frac{1}{2} g^{ij} p_i p_j$, where $\vec{p} = (p_1, p_2)$ are the momenta and $|\cdot|$ is the norm induced by g .

We say that a function $F : T^*S \rightarrow \mathbb{R}$ is an *integral* of the geodesic flow of g *cubic in momenta* (shortly: *cubic integral* for the metric g), if

- (1) in the local coordinates $x := x^1, y := x^2$ on the surface and the corresponding momenta p_x, p_y , F is a homogeneous polynomial in the momenta of degree 3:

$$F(x, y; p_x, p_y) = a_0(x, y) p_x^3 + a_1(x, y) p_x^2 p_y + a_2(x, y) p_x p_y^2 + a_3(x, y) p_y^3, \quad (1.1)$$

- (2) F is an integral of the geodesic flow of g , i.e., $\{H, F\} = 0$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket on T^*S .

Cubic integrals allow to construct different geometric structures on the surface; we shall use the so called *holomorphic Birkhoff-Kolokoltsov 3-codifferential*. We consider the complex structure on S given by g : a local complex coordinate $z = x + iy$ is determined by the property that the metric g has the isothermic form

$$g = \lambda(x, y)(dx^2 + dy^2) = \lambda(z, \bar{z}) dz d\bar{z}. \quad (1.2)$$

Then, the Birkhoff-Kolokoltsov 3-codifferential (corresponding to the cubic integral (1.1)) is a complex-valued symmetric $(3, 0)$ -tensor given by

$$A(z) := \left((a_0(x, y) - a_2(x, y)) + i(a_1(x, y) - a_3(x, y)) \right) \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z}. \quad (1.3)$$

As it was shown by Birkhoff [Bi], the coefficient $(a_0 - a_2) + i(a_1 - a_3)$ is a holomorphic function of the complex coordinate $z = x + iy$. Kolokoltsov observed in [Kol] that under a holomorphic (\Leftrightarrow conformal) coordinate change this coefficient changes as the corresponding coefficient of a complex $(3, 0)$ -tensor. Below in §2.2 we shall prove both properties of A and

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give several useful formulas for A in arbitrary (not necessarily isothermic) coordinates, such as how to compute A from F and how to check the holomorphicity of A , see Lemma 5.1.

The goal of this article is to give an answer to the following question:

Given a metric g and a holomorphic 3-codifferential A , how to decide whether there exists a cubic integral for the geodesic flow of g whose Birkhoff-Kolokoltsov 3-codifferential coincides with A ?

If such a cubic integral F exists, we say that A is *compatible with g* .

A *complete algorithmic answer* is given in §1.3 using Theorems 1.1, 1.2 and also Propositions 3.1, 4.2. Moreover, in the most interesting case, Theorems 1.1, 1.2 give an explicit formula for the cubic integral.

1.2. Main theorems. In the formulas below we use Einstein's summation convention, i.e., we sum over repeating upper and lower indices. The notation $a_{,jk}$ and so on means covariant derivation(s) of a function (or a tensor) a , and $a^{(ijk)}$ denotes the symmetrisation in indices i, j, k .

Let $x = x^1, y = x^2$ be arbitrary coordinates on the surface S . Given a metric $g = g_{ij}dx^jdx^i$ on S , let R_{ijk}^i be its Riemannian curvature, and set $\lambda(x, y) := \sqrt{\det(g_{ij})}$. Since our considerations are local, we equip S with the orientation given by the metric volume form $\omega_g = \lambda dx \wedge dy$.

We denote by $\{f, h\}_g := \frac{1}{\lambda} \left(\frac{\partial f}{\partial x} \frac{\partial h}{\partial y} - \frac{\partial h}{\partial x} \frac{\partial f}{\partial y} \right)$ the Poisson bracket of functions f, h on S with respect to ω_g . Further, the tensor $J^i_j := g^{ik}\omega_{kj}$ is the operator of rotation by 90° , it defines the complex structure on S compatible with the metric g and the orientation, i.e., corresponds to the multiplication by i in the complex coordinate $z = x + iy$.

Define the following smooth functions $\varphi_0, \dots, \varphi_3$, which are invariant algebraic expressions of the components of g and its derivatives:

- The Gauss curvature, which is the half of scalar curvature:

$$\varphi_0 := R := \frac{1}{2} R_{jik}^i g^{jk}; \quad (1.4)$$

- Half-square of the gradient of $\varphi_0 = R$, half-square of the gradient of the result:

$$\varphi_1 := \frac{1}{2} |\nabla \varphi_0|^2 = \frac{1}{2} g^{ij} \frac{\partial \varphi_0}{\partial x^i} \frac{\partial \varphi_0}{\partial x^j}; \quad \varphi_3 := \frac{1}{2} |\nabla \varphi_1|^2 = \frac{1}{2} g^{ij} \frac{\partial \varphi_1}{\partial x^i} \frac{\partial \varphi_1}{\partial x^j} \quad (1.5)$$

- The Poisson bracket of φ_0, φ_1 with respect to ω_g :

$$\varphi_2 := \{\varphi_0, \varphi_1\}_g := \frac{1}{\lambda} \left(\frac{\partial \varphi_0}{\partial x} \frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_0}{\partial y} \frac{\partial \varphi_1}{\partial x} \right). \quad (1.6)$$

Further, let A be a complex 3-codifferential on S given in *some* (not necessarily isothermic) coordinates (x^1, x^2) as symmetric $(3, 0)$ -tensor A^{ijk} with complex components. Define $\hat{A} := \Re(A)$ to be its real part, or $\hat{A}^{ijk} := \frac{1}{2}(A^{ijk} + \bar{A}^{ijk})$ on the level of components where \bar{a} means complex conjugation. Then $\hat{A} = (\hat{A}^{ijk})$ is a (usual) symmetric $(3, 0)$ -tensor with real components. The formula for the imaginary part $\Im(A^{ijk})$ is given in Section 5.

In addition to $\varphi_0, \dots, \varphi_3$, we define the functions $D_0, \dots, D_3, \mathcal{G}_2, \mathcal{G}_3$, 1-form $\mathcal{K} = \mathcal{K}_i dx^i$, and symmetric $(3, 0)$ -tensors \hat{B}^{ijk}, F^{ijk} . They all are certain invariant algebraic expressions

in g , \hat{A} , and their derivatives.

$$D_0 := 4 \mathfrak{R}(A^{ijk}{}_{;ijk}) = 4 \hat{A}^{ijk}{}_{;ijk}; \quad (1.7)$$

$$D_1 := g^{ij}(D_0)_{;i}(\varphi_0)_{;j} - 4 \hat{A}^{ijk}{}_{;i} \cdot (\varphi_0)_{;j} \cdot (\varphi_0)_{;k}; \quad (1.8)$$

$$D_2 := \{D_0, \varphi_1\}_g + \{\varphi_0, D_1\}_g; \quad (1.9)$$

$$D_3 := g^{ij}(D_1)_{;i}(\varphi_1)_{;j} - 4 \hat{A}^{ijk}{}_{;i} \cdot (\varphi_1)_{;j} \cdot (\varphi_1)_{;k} \quad (1.10)$$

All these functions have a clear geometric sense:

- D_0 is the triple divergence of $4 \cdot \hat{A}$;
- $g^{ij}(D_0)_{;i}(\varphi_0)_{;j}$ is the scalar product $\langle \nabla D_0, \nabla \varphi_0 \rangle$ of gradients,
- $\hat{A}^{ijk}{}_{;i}(\varphi_0)_{;j}(\varphi_0)_{;k}$ is the evaluation of the quadratic form corresponding to symmetric $(2, 0)$ -tensor $(\operatorname{div} \hat{A})^{jk} := \hat{A}^{ijk}{}_{;i}$ on the 1-form $d\varphi_0 = dR$;
- We obtain φ_3 from φ_1 by the same formula as φ_1 from φ_0 , similarly, we obtain D_3 from D_1 and φ_1 by the same formula as D_1 from D_0 and φ_0 .

Finally, define

$$\mathcal{G}_2 := \frac{1}{\lambda} \det \begin{pmatrix} (\varphi_0)_{;x} & (\varphi_0)_{;y} & D_0 \\ (\varphi_1)_{;x} & (\varphi_1)_{;y} & D_1 \\ (\varphi_3)_{;x} & (\varphi_3)_{;y} & D_3 \end{pmatrix} \quad \mathcal{G}_3 := \frac{1}{\lambda} \det \begin{pmatrix} (\varphi_0)_{;x} & (\varphi_0)_{;y} & D_0 \\ (\varphi_1)_{;x} & (\varphi_1)_{;y} & D_1 \\ (\varphi_2)_{;x} & (\varphi_2)_{;y} & D_2 \end{pmatrix}, \quad (1.11)$$

$$\mathcal{K}_i := \frac{1}{\varphi_2} \det \begin{pmatrix} (\varphi_0)_{;i} & D_0 \\ (\varphi_1)_{;i} & D_1 \end{pmatrix} \quad \hat{B}^{ijk} := g^{(ij} g^{k)l} J^m{}_l \mathcal{K}_m \quad F^{ijk} := \hat{A}^{ijl} + \hat{B}^{ijk}. \quad (1.12)$$

Theorem 1.1. *In the above notation, assume that the 3-codifferential A is holomorphic and that φ_2 is non-vanishing on S . Then there exists a cubic integral F with the Birkhoff-Kolokoltsov 3-codifferential A if and only if g and A satisfy the equations $\mathcal{G}_2 = 0$, $\mathcal{G}_3 = 0$.*

Moreover, such F is unique and in local coordinates (x^1, x^2) with the dual momenta (p_1, p_2) the integral F is given by

$$F(x^1, x^2; p_1, p_2) := F^{ijk}(x^1, x^2) p_i p_j p_k$$

where the symmetric $(3, 0)$ -tensor F^{ijk} is computed using the formulas (1.4)–(1.12).

Remarks. 1. Our definitions and formulas involve the orientation on the surface S , however, the final result is, of course, independent of the orientation. More precisely, after reversing the orientation every real expression (formula) either remains unchanged, or inverts the sign.

2. After appropriate “cosmetic” changes, Theorems 1.1, 1.2 remain valid also in the pseudo-Riemannian case, see Proposition 6.2.

Let us now consider the degenerate case when $\varphi_2 = \{R, \frac{1}{2}|\nabla R|^2\}$ vanishes identically. In this case we set $\varphi_0^* := \varphi_0$, $D_0^* := D_0$

$$\varphi_1^* := \Delta R \quad D_1^* := \Delta D_0 - 2 \mathfrak{R}((A_{;z} R_{;z})_{;z}). \quad (1.13)$$

and define the $*$ -versions φ_2^* , φ_3^* , \mathcal{G}_2^* , D_2^* , D_3^* , \mathcal{G}_3^* , \mathcal{K}_1^* , \mathcal{K}_2^* ... of the previous expressions by the replacing $\varphi_0, D_0, \varphi_1, \dots$ by $\varphi_0^*, D_0^*, \varphi_1^*, \dots$ in the formulas (1.6), (1.9), (1.10), (1.11), (1.12) above, see also Section 4. Further, let \mathcal{D} be that of two expressions $\mathcal{D}_i := \det \begin{pmatrix} (\varphi_0)_{;x^i} & D_0 \\ (\varphi_1)_{;x^i} & D_1 \end{pmatrix}$ (non- $*$ -versions!) for which $(\varphi_0)_{;x^i}$ is non-vanishing (any of two if both $(\varphi_0)_{;x}, (\varphi_0)_{;y}$ are non-zero.)

Theorem 1.2. *Let g be a metric and A a holomorphic 3-codifferential. Assume that the function $\varphi_2 = \{R, \frac{1}{2}|\nabla R|_g^2\}_g$ vanishes identically and the function $\varphi_2^* = \{R, \Delta_g R\}_g$ is non-vanishing.*

Then g admits a cubic integral F with the Birkhoff-Kolokoltsov 3-codifferential A if and only if g satisfies the PDEs \mathcal{G}_2^ , \mathcal{G}_3^* , and \mathcal{D} . Moreover, such F is unique and in local coordinates (x^1, x^2) with the dual momenta (p_1, p_2) the integral F is given by*

$$F(x^1, x^2; p_1, p_2) := F^{ijk}(x^1, x^2)p_i p_j p_k$$

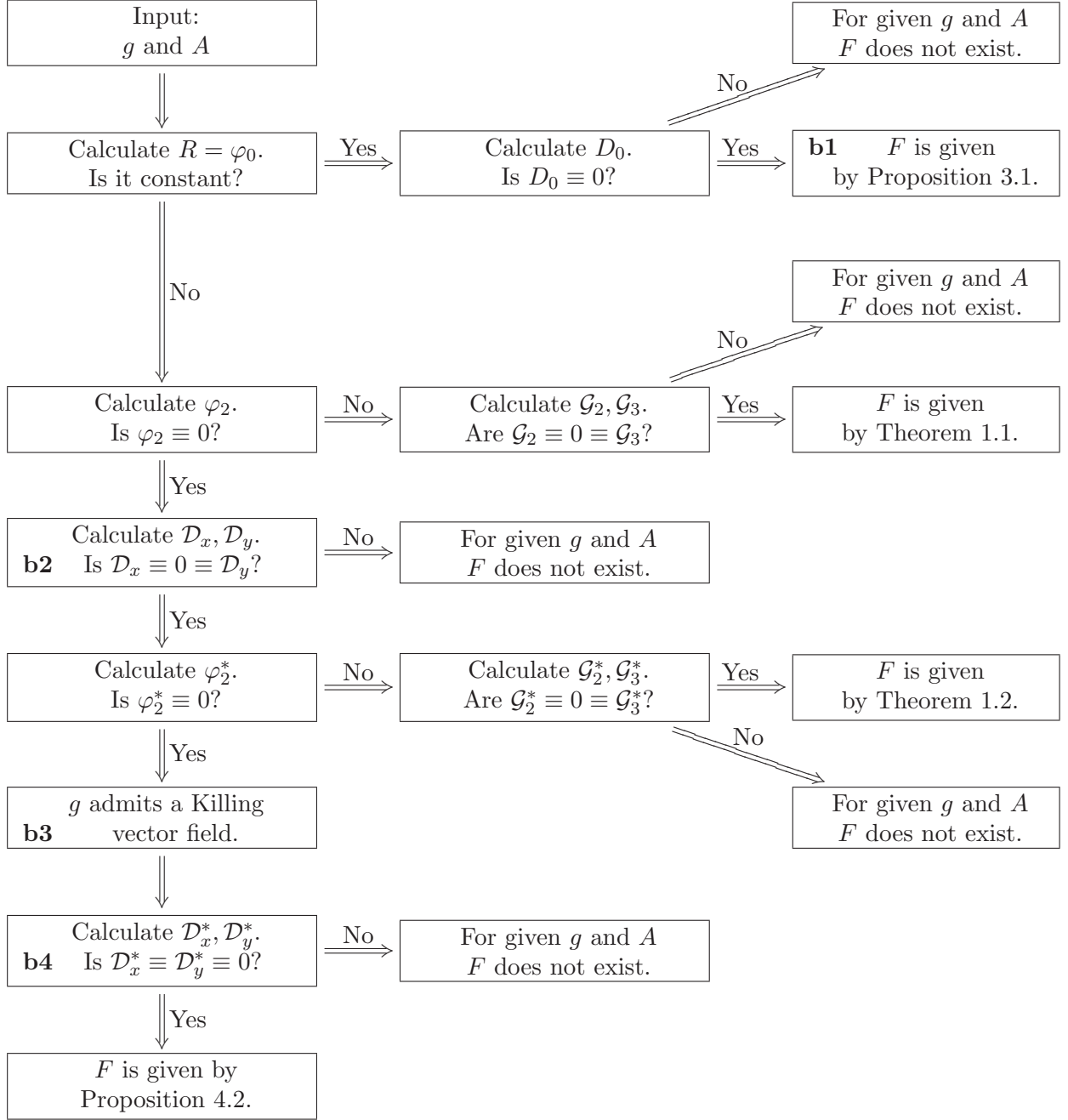
where the symmetric $(3, 0)$ -tensor F^{ijk} is computed by the $*$ -version of the formula (1.12), namely,

$$\mathcal{K}_i^* := \frac{1}{\varphi_2^*} \det \begin{pmatrix} (\varphi_0)_{;i} & D_0 \\ (\varphi_1^*)_{;i} & D_1^* \end{pmatrix} \quad \hat{B}^{ijk} := g^{(ij} g^{k)l} J^m_l \mathcal{K}_m^* \quad F^{ijk} := \hat{A}^{ijl} + \hat{B}^{ijk}. \quad (1.14)$$

Remark. The equations \mathcal{G}_2^* , \mathcal{G}_3^* are (always) necessary conditions for compatibility of A with g , but *not sufficient* in general. Nevertheless, if A is compatible with g and φ_2^* is non-vanishing, then the formulas (1.14) are valid and give the correct value of the cubic integral F , even in the case when φ_2 is also non-zero.

The only two cases which are not covered by Theorems 1.1, 1.2 are the case $\varphi_0 \equiv \text{const}$ (i.e., the metric has constant curvature) and the case $\varphi_2 \equiv 0$, $\varphi_2^* \equiv 0$. These cases are easier, and will be solved in Propositions 3.1, 4.2.

1.3. Algorithm for checking the existence of cubic integrals. Let g be a metric on a surface S and A a holomorphic 3-codifferential. Theorems 1.1, 1.2, classical results of Darboux and Eisenhart which we recall in §1.5, and also Propositions 3.1, 4.2 give us the following algorithm to decide whether there exists a cubic integral F with the given Birkhoff-Kolokoltsov 3-codifferential A . Moreover, in the most interesting cases covered by Theorems 1.1, 1.2 we obtain an explicit formula for F as algebraic expression in g , A and their derivatives. In other cases covered by Propositions 3.1, 4.2, the formula for F requires a solution of linear systems of ODEs. Corresponding calculations can be easily realized using computing software such as Maple[®] or Mathematica[®].



Let us comment some boxes in the flowchart of the algorithm. Recall that A is *compatible with* g if A is the Birkhoff-Kolokoltsov 3-codifferentials associated to some cubic integral F of the metric g .

- (b1) If R is constant, then every cubic integral is of the form $F = \alpha^{ijk} L_i L_j L_k$ with constants $\alpha^{ijk} \in \mathbb{R}$ where L_i , $i = 1, 2, 3$ are three linear (polynomial of degree 1) invariants which are linearly independent. Thus the cubic integrals form a linear space of dimension 10, see [Kr]. On the other hand, the space of holomorphic 3-codifferentials A has infinite dimension. Proposition 3.1 shows that the equation $D_0 = 0$ is necessary and sufficient condition for compatibility of A with g and, if it is fulfilled, the space of cubic integrals depends on 3 real parameters. More precisely, the a generic cubic integral has the form $\mathfrak{R}(A) + b^i p_i \cdot H$ where $\vec{b} = (b^i)$ is a vector field with given values $\vec{b}(P)$ and $\text{rot } \vec{b}(P)$ at a

given point $P \in S$. The construction of the vector field $\vec{b} = (b^i)$ reduces to solution of certain linear systems of ODEs.

- (b2) It is sufficient to calculate only one of the expressions $\mathcal{D}_x, \mathcal{D}_y$, namely one such the corresponding $\frac{\partial \varphi_0}{\partial x}, \frac{\partial \varphi_0}{\partial y}$ is not zero, see Theorem 1.2.
- (b3) By a classical result of Darboux and Eisenhart (see discussion in §1.5) the vanishing of both φ_2 and φ_2^* is equivalent to the local existence of a Killing vector field which we denote by ξ . The latter is equivalent to the existence of a linear integral which we denote by L . Since Lie derivative of the curvature R along ξ must vanish, we have that ξ is proportional to the Hamiltonian vector field of R , $\xi = f(x, y) \cdot \vec{X}_R$. Such $f(x, y)$ is unique up to a constant factor, and can be found from the PDE-system $\mathcal{L}_{f\vec{X}_R}g = 0$ on the unknown function f . Note that the PDE-system $\mathcal{L}_{f\vec{X}_R}g = 0$ is Frobenius and, in view of conditions $\varphi_2 = 0$ and $\varphi_2^* = 0$, is in involution, i.e., is essentially a system of ODE.
- (b4) The expressions $\mathcal{D}_x^*, \mathcal{D}_y^*$ and the part of the algorithm starting from the box (b3) is treated in §4.2.

In the pseudo-Riemannian this algorithm basically works, however there exists an important difference. Namely, the condition $|\nabla R|_g^2 = 0$ does not imply that $R = \text{const}$. As a consequence we obtain a new special class of pseudo-Riemannian metrics whose counterpart does not exist in the Riemannian case. They are not metrics of constant curvature, they have $\varphi_2 \equiv 0 \equiv \varphi_2^*$, however, as we prove in Proposition 6.3, the metrics from this class admit no cubic integrals, and therefore no Killing vector fields. We refer to Section 6 for further details and description of an additional step in the algorithm.

We emphasise here that the algorithm works in any (not necessary isothermal) coordinate system. In particular, Lemma 5.1 gives a way how to verify whether the data (g, A) have sense, i.e., whether *real* (3,0)-tensor $\hat{A} = \Re(A)$ corresponds to the real part of a holomorphic 3-codifferential.

1.4. Motivation and history. The importance of polynomial integrals for studying the geodesic flow was recognised long time ago. Indeed, it was Jacobi who had realised that the geodesic flow on the ellipsoid admits an “extra” quadratic integral. This allowed Jacobi to integrate the geodesic flow on the ellipsoid.

Polynomial integrals for the geodesic flow are interesting for physics and for differential geometry. The interest from physics is due to the following observation (called the Maupertuis principle, or coupling constant transform): for sufficiently large energy levels h the restriction of the Hamiltonian system on T^*M with the Hamiltonian $H := \frac{1}{2}|\vec{p}|^2 + V$ (where $\frac{1}{2}|\vec{p}|^2$ is again the kinetic energy corresponding to g , and $V : S \rightarrow \mathbb{R}$ is the potential) to the energy surface $\{(x, \vec{p}) \in T^*M \mid H(x, \vec{p}) = h\}$ has the same unparametrised orbits as the geodesic flow of the metric $g_h := (h - V)g$. Moreover, if the Hamiltonian system is polynomially-integrable, then the geodesic flow is also polynomially-integrable. More precisely, if a function

$$F(x, y; p_x, p_y) = \underbrace{a_0 p_x^3 + a_1 p_x^2 p_y + a_2 p_x p_y^2 + a_3 p_y^3}_{F_3(x, y; p_x, p_y)} + \underbrace{c_0 p_x + c_1 p_y}_{F_1(x, y; p_x, p_y)} \quad (1.15)$$

is an integral of the Hamiltonian system with the Hamiltonian $\frac{1}{2}|\vec{p}|^2 + V$, then the (homogeneous cubic in momenta) function

$$F_h := F_3 + H_h \cdot F_1 \quad (1.16)$$

is an integral for the Hamiltonian system with the Hamiltonian $H_h := \frac{1}{2(h-V)}|\vec{p}|^2$, i.e., for the geodesic flow of the metric $g_h := (h - V)g$.

Note that the assumption that the integral (1.15) does not contain the terms F_0 and F_2 of degree 0 and 2 in momenta is a natural one: by Whittaker [Wh], if a function of the form $F_0 + F_1 + F_2 + F_3$ is an integral for the Hamiltonian system with the Hamiltonian $H := \frac{1}{2}|\vec{p}|^2 + V$, then $F_1 + F_3$ is also an integral.

Metrics admitting integrals polynomial in velocities are also interesting for geometry, since the existence of such integrals implies interesting geometric properties of the metric. Moreover, study of polynomial integrals helps to solve pure geometrical problems. For example, the existence of the integral of degree 1 is equivalent to the existence of a Killing vector field, and implies the existence of the coordinate system such that the component of the metric does not depend on one of the coordinates [Da]. The existence of an integral of degree 2 is equivalent to the existence of another metric having the same geodesic with the initial one [Ma-To1, Ma-To2]. The last observation was recently used [Br-Ma-Ma, Ma] in the solution of two geometric problems explicitly formulated by Sophus Lie in 1882.

It is not clear whether the existence of the integral of degree 3 has such a clear geometric meaning, but still, as we mentioned above, it allows one to construct a holomorphic 3-codifferential with interesting properties, and, as it was shown in [Du-Ma-To], a volume-preserving vector field on the surface.

1.5. Previous results in this field. The most classical result in this direction is in folklore attributed to Bonnet, and appeared in the books of Darboux [Da] and of Eisenhart [Ei].

Given a metric $g = (g_{ij}(x, y))$ on a surface, let us consider the functions $\varphi_0, \varphi_1, \varphi_1^*, \varphi_2, \varphi_2^*$ from §1.1: $\varphi_0 := R$ is the Gauss curvature, $\varphi_1 := \frac{1}{2}|\nabla R|_g^2$ is half-square of the gradient of R , $\varphi_1^* := \Delta_g R = \Delta_g \varphi_0$ the metric Laplacian of $R = \varphi_0$, and $\varphi_2 := \{\varphi_0, \varphi_1\}_g$, $\varphi_2^* := \{\varphi_0, \varphi_1^*\}$ are the Poisson brackets.

One can show (see Darboux [Da, §§688, 689] and Eisenhart [Ei, pp. 323-325]) that the metric admits a nontrivial integral linear in momenta (locally, in a neighbourhood of almost every point) if and only if the invariants φ_2, φ_2^* vanish.

Though Darboux and Eisenhart proved this result for Riemannian case only, their proof can be generalised to the pseudo-Riemannian case with the following reformulation of the assertion: *if the equations $\varphi_2 = 0$ and $\varphi_2^* = 0$ are satisfied and $\frac{1}{2}|\nabla R|_g^2 = \varphi_1$ is non-zero, then g admits a nontrivial linear integral.* We study the class of pseudo-Riemannian metrics with the condition $|\nabla R|_g^2 \equiv 0$ in Section 6, and prove that they admit a linear integral iff $R = \text{const}$.

We see that we have an algorithmic way to decide whether a given metric admits a linear integral of degree: one need to calculate the invariants φ_2, φ_2^* which are algebraic expressions in the components of the metric and their derivatives up to order 5, and compare them with zero.

It was a long standing problem to find a similar algorithmic way to decide whether a given metric admits an integral quadratic in momenta. The first attempts are due to Roger Liouville¹ [Li] and Koenigs [Koe]. They understood that the PDE-system for the coefficients of the integral is linear and of finite type, implying that that there must exist an algorithmic

¹Roger Liouville was a younger relative of the more famous Joseph Liouville and attended his lectures at the Ecole Polytechnique

way to decide whether a given metric admits an integral quadratic in momenta. Unfortunately, the calculation were too complicated to be fulfilled in that time. An interesting approach to this problem is due to Sulikovski [Su]: he invented a trick which allowed to simplify the calculation. Finally, the problem was solved in recent independent works by Kruglikov [Kr] and Bryant et al [Br-Du-Ea].

Geodesic flows admitting cubic integrals is a much more difficult subject than those admitting linear or quadratic integrals. Indeed, metric admitting linear and quadratic integrals are completely described: the local description is done by classics [Da, §§540–596], see [Bo-Ma-Pu1, Bo-Ma-Pu2] for the discussion of the pseudo-Riemannian case, and the global (= if the surface is closed) is due to Kolokoltsov [Kol] and Kiyohara [Ki1], see the survey [Bo-Ma-Fo]. The metrics admitting linear integrals are parameterised by one function of one variable, the metrics admitting quadratic integrals are parameterised by two functions of one variable. No such description is known in the cubic case (though from general theory of PDE it follows that locally, in the analytic formal category, the metrics admitting cubic integrals are parametrised by 3 functions of one variable).

Though the complete description of metrics admitting cubic integrals is not known yet, there exists a lot of local examples (the first local examples appeared already in Darboux [Da, §619]). Some of most interesting examples come from physics, see the survey [Hi], or are motivated by physics [Ha, Ye, Ve-Ts].

It appears to be difficult to construct metrics admitting cubic integrals on closed surfaces. As it was shown by Kozlov [Koz] and Kolokoltsov [Kol], the surfaces of genus ≥ 2 do not admit (nontrivial) polynomial integrals; we repeat the argument of Kolokoltsov in the proof of Corollary 2.1. Of course, the metrics admitting linear integral also admit cubic integrals, since the cube of the linear integral is a cubic integral. Besides these half-trivial examples, the list of known examples is very short: there is a series of explicit examples coming from Goryachev case of rigid body motion and its generalisation, a series of recently found examples due to [Du-Ma], and proof of the existence of other examples in [Ki2], [Se].

In particular, in all known examples the surface is the sphere (i.e., there is no known example of metric on the torus admitting cubic integral and do not admitting linear integral; moreover, Bolsinov et al conjectured [Bo-Ko-Fo] that such examples can not exist).

The construction of curvature-type invariants whose vanishing is equivalent to the existence of a cubic integral is discussed by Kruglikov in [Kr, §12]. In particular, he has shown that the dimension of the space of cubic integrals is ≤ 10 , moreover, the metrics admitting 10-dimensional space of cubic integrals have constant curvature. It was conjectured by Kruglikov in [Kr] that the next largest value of the dimension of the space of cubic integrals is 4 and that in this case g is Darboux superintegrable (the definition see in [K-K-M-W] or in [Br-Ma-Ma, §2.2.4]). If the Kruglikov conjecture is true, [Kr, Theorem 2] gives curvature-type invariants whose vanishing is equivalent to the existence of the 4-dimensional space of cubic integrals.

In the present paper, we answer the question whether the pair (metric g , holomorphic 3-codifferential A) is compatible in the sense there exists a cubic integral for g whose Birkhoff-Kolokoltsov form is A . On the level on PDE, we reduce the number of unknown functions, which of course make the problem easier. This is the reason that our answer (for cubic integrals) is more simpler than the answers of [Kr] and [Br-Du-Ea] (for quadratic integrals). From other side, the assumption that A is given is natural from the viewpoint of classical mechanics. Indeed, all integrals F_h corresponding to different values of h (see §1.4) have the

same holomorphic 3-codifferential A . Moreover, on closed surfaces, the space of holomorphic 3-codifferentials is finite-dimensional. That means that, in the investigation of cubic integrals on closed surfaces, one can view our compatibility conditions as the conditions on the metric only depending on finitely many parameters.

2. COMPLEX CALCULUS AND PROLONGATION-PROJECTION METHOD.

2.1. Homogeneous polynomial integrals. Let g be a metric on a surface S and let $H = \frac{1}{2}|p|_g^2$ be a Hamiltonian given by the kinetic energy and let $F(x, p)$ be its integral which is polynomial of degree $\leq d$ in momenta. Then $F(x, p) = \sum_{j=0}^d F_j(x, p)$ where each component F_j is homogeneous of degree j in momenta. Direct computation shows that each Poisson bracket $\{H, F_j\}$ is homogeneous of degree $j + 1$ in momenta. The uniqueness of the decomposition of a polynomial into homogeneous components implies that the Poisson bracket $\{H, F\}$ vanishes if and only if each bracket $\{H, F_j\}$ vanishes. By this reason, considering polynomial integrals of purely kinetic Hamiltonians $H = \frac{1}{2}|p|_g^2$ we can restrict ourselves to homogeneous polynomial integrals F .

2.2. Birkhoff-Kolokoltsov codifferential. Every metric g on a surface S admits locally so-called *isothermic coordinates* (x, y) in which the metric has the form $g = \lambda(x, y)(dx^2 + dy^2)$. If we orient S by means of these coordinates, then the function $z := x + iy$ will be a complex coordinate for the *induced complex structure* on S , characterised by the following property: The multiplication of z by i is the rotation of the chart (x, y) by 90° . The existence of isothermic coordinates was discovered by B. Riemann. A detailed proof under very weak assumption on the metric (e.g., g is merely continuous) can be found in [Beg].

In forthcoming calculus we use such a complex coordinate z and its conjugate \bar{z} instead of real coordinates (x, y) , since most formulas become simpler and more compact. The corresponding dual coordinates are $p := \frac{1}{2}(p_x - ip_y)$ and $\bar{p} := \frac{1}{2}(p_x + ip_y)$, the function H is expressed as $H = \frac{2p\bar{p}}{\lambda}$, and the canonical symplectic form in the cotangent bundle T^*S (“phase space”) is expressed as $\omega_{can} = dp \wedge dz + d\bar{p} \wedge d\bar{z}$.

Now assume that $F = F(z, \bar{z}, p, \bar{p})$ is a (local) integral of H cubic in momenta. Then $F = \Re(a \cdot p^3 + b \cdot p^2\bar{p})$ where \Re denotes the real part of a complex expression and $a = a(z, \bar{z})$, $b = b(z, \bar{z})$ are some smooth (local) *complex-valued* functions on S . An explicit calculation (see [Du-Ma-To]) yields:

$$\{F, H\} = \frac{2}{\lambda^2} \Re(p^4 \cdot a_{\bar{z}} + p^3 \bar{p} \cdot (b\lambda_{\bar{z}} + 3a\lambda_z + \lambda b_{\bar{z}} + \lambda a_z) + p^2 \bar{p}^2 \cdot (2b\lambda_z + \lambda b_z)), \quad (2.1)$$

where $(\cdot)_z$ and $(\cdot)_{\bar{z}}$ denote the complex derivatives:

$$f_z = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad f_{\bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

Since the term $p^4 \cdot a_{\bar{z}}$ must vanish, we obtain immediately the following

Lemma 2.1. *In the situation above, if $F = \Re(a \cdot p^3 + b \cdot p^2\bar{p})$ is an integral, then the coefficient a is a holomorphic.*

Rewrite any function $F(x^1, x^2; p_1, p_2)$ cubic in momenta as a polynomial $F = F^{ijk} p_i p_j p_k$ in p_i . Then, under change of coordinates, its coefficients $F^{ijk}(x^1, x^2)$ transform as components of a symmetric $(3, 0)$ -tensor. The same holds for the “ a -component” of F : Under *complex*

(\Leftrightarrow holomorphic) coordinate change $z \mapsto z'$ it transforms as $a' = a \cdot \left(\frac{\partial z'}{\partial z}\right)^3$. Indeed, the momenta p and \bar{p} are transformed as $p = p' \frac{dz'}{dz}$, $\bar{p} = \bar{p}' \frac{d\bar{z}'}{d\bar{z}}$ implying

$$F = \Re(a \cdot p^3 + b \cdot p^2 \bar{p}) = \Re\left(\underbrace{a \cdot \left(\frac{dz'}{dz}\right)^3}_{a'} \cdot p'^3 + b \cdot \left(\frac{dz'}{dz}\right)^2 \frac{d\bar{z}'}{d\bar{z}} \cdot p'^2 \bar{p}'\right).$$

This means that $A := a(z) \left(\frac{\partial}{\partial z}\right)^{\otimes 3}$ is a well-defined section of the bundle $(T^{(1,0)}S)^{\otimes 3}$ independent of the choice of a local holomorphic coordinate. By Lemma 2.1, the section is holomorphic.

Definition 2.1. For any function $F(x^1, x^2; p_1, p_2)$ cubic in momenta the complex tensor $A := a \cdot \left(\frac{\partial}{\partial z}\right)^{\otimes 3}$ is called the *Birkhoff-Kolokoltsov 3-codifferential* associated with F . We shall usually denote by $a = a(x, y) = a(z, \bar{z})$ its complex-valued coefficient (wrt. some complex coordinate z), and denote by $\hat{A} := \Re(A)$ its real part. The latter is a symmetric $(3, 0)$ -tensor.

Since every non-vanishing holomorphic function $f(z)$ in one variable has isolated zeroes, the same is true for any holomorphic 3-codifferential. Notice that if $a(z)$ is holomorphic and non-vanishing, then locally near a given point p there exists the root $\sqrt[3]{a(z)}$ which is again a non-vanishing holomorphic function. It follows that making the change to the new complex coordinate z' given by $z' := \int \frac{dz}{\sqrt[3]{a(z)}}$, the component $a(z)$ transforms into the constant $a'(z') \equiv 1$.

Another applications of holomorphicity of A are as follows:

Corollary 2.1 ([Kol]). *i) No Riemannian metric on a closed oriented surface S of genus $g \geq 2$ admits a global non-trivial polynomial integral F .*

ii) Let S be the 2-torus T^2 , g a metric, z a linear holomorphic coordinate, and $F = \Re(a \cdot p^3 + b \cdot p^2 \bar{p})$ a cubic integral. Then a is constant.

Proof. *i)* As is shown in §2.1, we may assume that our invariant F is homogeneous of degree k . Evidently one can take F in the form $F(z, \bar{z}, p, \bar{p}) = \Re(a \cdot p^k) + p\bar{p} \cdot B$ where $B(z, \bar{z}, p, \bar{p})$ is homogeneous in momenta of degree $k-2$. An explicit computation shows that similarly to the cubic case $\{H, F\} = \frac{2}{\lambda^2} \Re(a_{\bar{z}} \cdot p^{k+1}) + p\bar{p} \cdot C$ for some function $C(z, \bar{z}, p, \bar{p})$ which is homogeneous polynomial of degree $k-1$ in momenta. Thus $a_{\bar{z}}$ must vanish which means that $a(z)$ is holomorphic as in the cubic case. Similarly to the cubic case, $A := a \cdot \left(\frac{\partial}{\partial z}\right)^k$ is a well-defined *holomorphic* section of the bundle $(T^{(1,0)}S)^{\otimes k}$.

The classical complex geometry (see e.g., [Gr-Ha]) says that for any non-zero *meromorphic* section A of the bundle $(T^{(1,0)}S)^{\otimes k}$ the difference $n(A) - p(A)$ between the numbers of zeroes and poles of A is $k(2-2g)$. Since A is holomorphic in our case, there are no poles and $k(2-2g)$ must be non-negative. Consequently, in the case $g \geq 2$ A must vanish identically. It follows that $\frac{1}{\lambda} B = F \cdot H^{-1}$ is a global homogeneous integral for H of degree $k-2$. Now we can apply induction in k .

ii) A local holomorphic coordinate on a torus T^2 is *linear* if it comes from the standard complex coordinate z on the complex plane \mathbb{C} under some conformal isomorphism $T^2 \cong \mathbb{C}/\Lambda$ for some 2-lattice $\Lambda \subset \mathbb{C}$. An equivalent condition is that $\frac{\partial}{\partial z}$ is a globally defined holomorphic vector field. Such a field on a torus is non-vanishing, and therefore a is a globally defined holomorphic function on a closed Riemann surface. Such a function must be constant by Liouville's theorem. \square

2.3. Prolongation-projection method. The “naive” idea behind this method is very simple. Suppose we have a system $\mathcal{S} = \{\mathcal{E}_i\}$, of PDEs which is *overdetermined*, i.e., the number of equations is bigger than the number of unknown functions. In our case, we have two equations on one unknown function. We *prolong* the system, i.e., we differentiate the equations with respect to all variables (in our case, x and y) and add the results to the system \mathcal{S} . In other words, we consider the systems

$$\mathcal{S}^{(1)} := \{\mathcal{E}_i\} \cup \left\{ \frac{\partial}{\partial x} \mathcal{E}_i \right\} \cup \left\{ \frac{\partial}{\partial y} \mathcal{E}_i \right\} \quad \text{— first prolongation,}$$

$$\mathcal{S}^{(2)} := \{\mathcal{E}_i\} \cup \left\{ \frac{\partial}{\partial x} \mathcal{E}_i \right\} \cup \left\{ \frac{\partial}{\partial y} \mathcal{E}_i \right\} \cup \left\{ \frac{\partial^2}{\partial x^2} \mathcal{E}_i \right\} \cup \left\{ \frac{\partial^2}{\partial x \partial y} \mathcal{E}_i \right\} \cup \left\{ \frac{\partial^2}{\partial y^2} \mathcal{E}_i \right\} \quad \text{— second prolongation,}$$

and so on. Obviously, the operation “prolongation” does not change the set of sufficiently smooth solutions of the system. Indeed, every (sufficiently smooth) solution of \mathcal{S} is also a solution of the prolonged systems. From the other side, every solution of the prolonged system is evidently a solution of \mathcal{S} , because \mathcal{S} is a part of the prolonged system.

If the system is overdetermined, then generically the number of new equations grows faster than the number of derivatives of the unknown functions. Then, in the generic case, after say l prolongations one can resolve the prolonged system $\mathcal{S}^{(l)}$ with respect to the highest derivatives. In this case, the system \mathcal{S} is called the systems of *finite type*. In our case, the system is indeed of finite type, and the number of needed prolongations is $l = 1$.

After this point, we can start the prolongation-*projection* procedure. The key observation here is as follows: The successive prolongation $\mathcal{S}^{(l+1)}$ can be still resolved with respect to the highest derivatives, the number of the equations in $\mathcal{S}^{(l+1)}$ will be generically greater than the number of highest derivatives, and with the help of algebraic manipulations we can obtain *new* equations of *lower* degree. This algebraic manipulation — expressing of higher derivatives via lower ones using part of equations and substituting these expression in remaining equations — is called *projection* procedure. We can repeat the prolongation-projection many times (clearly, is sufficient to differentiate only *newly obtained* equations).

In the generic case, after finite number of prolongation-projections, we obtain a system in which the number of *algebraically independent* equations is greater than the number of partial derivatives of unknown functions (we consider the unknown functions as partial derivatives of order 0). Such an algebraic system is inconsistent. In this case the initial system \mathcal{S} has no solution. It may also happen that certain prolongation $\mathcal{S}^{(k)}$, considered as a system of algebraic equations on the derivatives of the unknown functions, is algebraically inconsistent, even if the number of algebraically independent is less than the number of partial derivatives.

Thus, the existence of a solution of an overdetermined system \mathcal{S} of finite type implies that after certain number of prolongations and prolongation-projections we come to the point where prolongation-projections do not produce essentially new equations. The latter means that the equations we obtain are algebraic corollaries of the equations we already have. Moreover, the system of algebraic equations on the partial derivatives of unknown functions is consistent. In this case, the system is called *involution*, or *in involution*. One can show that (under certain additional assumptions which are fulfilled in our case) involutive systems can be solved locally. Moreover, one can find all solutions with the help of solving of certain ODEs and of algebraic operations.

As we noted above, a generic overdetermined system of finite type is inconsistent and can not be put to be in involution by prolongation-projections. One obtains a system in

involution after certain number of prolongations and prolongation-projections, if the coefficients of the initial system satisfies certain partial differential equations (called *integrability conditions*). These PDE on the coefficients are equivalent to two conditions:

- (1) we obtain no new equations after certain prolongation-projection.
- (2) All systems we obtain by prolongations and prolongation-projections, considered as algebraic systems on the partial derivatives of the unknown functions, are algebraic consistent.

In our paper, we find explicitly all integrability conditions for the system that corresponds that g and A are compatible: they are $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_2^*, \mathcal{G}_3^*, \mathcal{D}_x, \mathcal{D}_y, D_0$ from Theorems 1.1, 1.2 and Propositions 3.1, 4.2.

The prolongation-projection method is a very powerful method for finding solutions of overdetermined systems of PDE of finite type. Moreover, in our case the initial system of PDE is an inhomogeneous linear system, implying that standard difficulties due to impossibility of solving explicitly systems of algebraic equations do not appear, so the method can be applied algorithmically. Actually, the only difficulty in applying the prolongation-projection method for linear overdetermined systems of PDE of finite type is the calculational one. In many cases, even modern computer algebra programs are not powerful enough for using prolongation-projection.

We overcome this difficulty with the help of “advanced complex calculus” introduced in §2.4: roughly speaking, this calculus is a collection of quite nontrivial tricks that allows us to ‘hide’ the covariant derivatives of the objects, and, therefore, makes all formulas very compact, so we could do prolongation-projections “by hands”, without using computer algebra. Moreover, all differential equations on the coefficient of the metric and of the Birkhoff-Kolokoltsov form are automatically expressed in the invariant form, so one can calculate them in an arbitrary coordinate system.

Notation. Applying the prolongation-projection method we use the following terminology. Let \mathcal{E} be a (differential) equation or expression and $\mathcal{S} = \{\mathcal{E}_i\}$ a system (ie., a set) of such equation. Then \mathcal{E} is an *algebraic consequence* of \mathcal{S} if it can be deduced from \mathcal{S} by means of algebraic manipulations, and a *differential consequence* if \mathcal{E} is an algebraic consequence of \mathcal{E} and partial derivatives of equations from \mathcal{S} . In this paper we work with *linear* PDEs and make linear-algebraic manipulation with equations. In particular, we have *linear-algebraic consequences* of linear systems of equations, and differential consequences can be obtained applying *linear* differential operators.

Remark. The name “prolongation-projection” is natural in the context of jet bundle geometry, we recommend the textbook [Kr-Ly-Vi] as a source. Equivalent approach is called Cartan, or Cartan-Kähler theory, we recommend the textbooks [B-C-G-G-G, Iv-La] for more details. The system of PDE we consider in our paper is easier than the generic systems treated in [Kr-Ly-Vi, B-C-G-G-G, Iv-La] (because they are linear and of finite type), that’s why we can achieve our goals without going too far into the geometric theory of PDE. Actually, our intention is to make the paper understandable for a standard mathematician working in mathematical physics; that’s why we avoided the terminology of the geometric theory of PDE.

2.4. Advanced complex calculus. Our next goal is to develop certain calculus which will allow us to simplify computation and express results and formulas in more compact form.

Let (S, g) be a Riemannian surface with a fixed orientation and $z = x + iy$ a local complex coordinate. Let $T^*S^{\mathbb{C}}$ be the complexified cotangent bundle, its sections are complex 1-forms on S . Then sections $dz = dx + idy$ and $d\bar{z} = dx - idy$ form a local frame of $T^*S^{\mathbb{C}}$. Consider the decomposition $T^*S^{\mathbb{C}} = \Omega^{(1,0)}S \oplus \Omega^{(0,1)}S$ in which the summands are generated by the forms dz and respectively $d\bar{z}$. The formula for transformations of the forms dz and $d\bar{z}$ under *holomorphic* coordinate changes shows that the decomposition $T^*S^{\mathbb{C}} = \Omega^{(1,0)}S \oplus \Omega^{(0,1)}S$ is independent of the choice of a local complex coordinate. A similar decomposition $TS^{\mathbb{C}} = T^{(1,0)}S \oplus T^{(0,1)}S$ for the complexified tangent bundle is obtained using local vector fields $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

We refer to [Gr-Ha, For] for basic properties of these decompositions. Main properties of the bundles $T^{(1,0)}S$, $T^{(0,1)}S$, $\Omega^{(1,0)}S$, $\Omega^{(0,1)}S$ which we shall use are as follows:

- (1) The bundles $\Omega^{(1,0)}S$ and $\Omega^{(0,1)}S$ are *complex dual line bundles* to $T^{(1,0)}S$ and $T^{(0,1)}S$, respectively.

Using this fact, we define complex line bundles $T^{[p,q]}S$ and $\Omega^{[p,q]}S$ with *integer* p, q setting

$$T^{[p,q]}S := (T^{(1,0)}S)^{\otimes p} \otimes (T^{(0,1)}S)^{\otimes q}, \quad \Omega^{[p,q]}S := (\Omega^{(1,0)}S)^{\otimes p} \otimes (\Omega^{(0,1)}S)^{\otimes q}.$$

These bundles should be not confused with bundles of tensors of type (p, q) used in differential geometry, and with bundles of (p, q) -forms used in complex analysis and geometry. In order to emphasise the difference, we use brackets $[p, q]$ instead of parentheses (p, q) .

- (1') From the definition and property (1) we obtain immediately

$$\Omega^{[p,q]}S = T^{[-p,-q]}S \quad \text{and} \quad T^{[p,q]}S \otimes T^{[p',q']}S = T^{[p+p',q+q']}S.$$

The bundle $T^{[0,0]}S = \Omega^{[0,0]}S$ is trivial and its sections are usual complex-valued functions.

- (2) The bundle $(T^*S)^{(\otimes k)} \otimes \mathbb{C}$ (respectively $(TS)^{(\otimes k)} \otimes \mathbb{C}$) splits into the sum of line bundles each isomorphic to some $\Omega^{[p,q]}S$ (respectively $T^{[p,q]}S$) with $p, q \geq 0$ and $p + q = k$. A similar splitting holds for a general complex tensor bundle $(TS)^{(\otimes k)} \otimes (T^*S)^{(\otimes l)} \otimes \mathbb{C}$. The components of $(T^*S)^{(\otimes k)} \otimes \mathbb{C}$ can be indexed by the order of factors $\Omega^{(1,0)}S$, $\Omega^{(0,1)}S$ in the product.

Notice that the subbundle of *symmetric tensors* of $(T^*S)^{(\otimes k)} \otimes \mathbb{C}$ is isomorphic to the sum $\bigoplus_{p=0}^k \Omega^{[p,k-p]}S$, so that each $\Omega^{[p,k-p]}S$ appears exactly once. In the special case $k = 2$ the power $(T^*S)^{(\otimes 2)} \otimes \mathbb{C}$ has as summands one bundle $\Omega^{[2,0]}$, one bundle $\Omega^{[0,2]}$, and two bundles $\Omega^{(1,0)}S \otimes \Omega^{(0,1)}S$, $\Omega^{(0,1)}S \otimes \Omega^{(1,0)}S$ each isomorphic to $\Omega^{[1,1]}$.

Moreover, for *positive* $i \leq \min(p, k)$, $j \leq \min(q, l)$, the natural isomorphism

$$T^{[p,q]}S \otimes \Omega^{[k,l]}S \xrightarrow{\cong} T^{[p-i,q-j]}S \otimes \Omega^{[k-i,l-j]}S \quad (2.2)$$

coincide with the $(i+j)$ -fold index contraction in the space of symmetric complex-valued $(p+q, k+l)$ -tensors.

If $z = x + iy$ is a local complex coordinate on S , then

$$dz \otimes d\bar{z} = (dx^2 + dy^2) + i(dx \otimes dy - dy \otimes dx) = (dx^2 + dy^2) + i dx \wedge dy$$

and $d\bar{z} \otimes dz = (dx^2 + dy^2) - i dx \wedge dy$. Thus the symmetric part of the sum $\Omega^{(1,0)}S \otimes \Omega^{(0,1)}S \oplus \Omega^{(0,1)}S \otimes \Omega^{(1,0)}S$ is spanned by $dz \bullet d\bar{z} := \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = dx^2 + dy^2$ and its antisymmetric part spanned by $dz \wedge d\bar{z} = 2i dx \wedge dy$. This gives another decomposition of sum $\Omega^{(1,0)}S \otimes \Omega^{(0,1)}S \oplus \Omega^{(0,1)}S \otimes \Omega^{(1,0)}S$ into two bundles each isomorphic to $\Omega^{[1,1]}$.

The anti-symmetric part is the bundle $\Lambda^2 S \otimes \mathbb{C}$ of complex 2-forms on S and the power $(T^*S)^{\otimes 2} \otimes \mathbb{C}$ splits into the sum of $\Omega^{[2,0]}$, $\Omega^{[0,2]}$, and $\Omega^{[1,1]}$

In what follows we shall use only symmetric tensors. Thus the product $dz d\bar{z}$ will be understood as the symmetric one. This corresponds to the embedding of $\Omega^{[1,1]}$ into $T^*S \otimes T^*S \otimes \mathbb{C}$ with symmetric image.

- (3) The complex conjugation in the bundles $(TS)^{\mathbb{C}}$ and $(T^*S)^{\mathbb{C}}$ induces conjugacy bundle homomorphisms $\sigma : T^{(1,0)}S \rightarrow T^{(0,1)}S$ and $\sigma : \Omega^{(1,0)}S \rightarrow \Omega^{(0,1)}S$ with $\sigma^2 = \text{Id}$.

This σ extends to conjugacy homomorphisms $\sigma : T^{[p,q]}S \rightarrow T^{[q,p]}S$ and $\sigma : \Omega^{[p,q]}S \rightarrow \Omega^{[q,p]}S$ for any pair of integers p, q . In the case $T^{[0,0]} = \Omega^{[0,0]} = \mathbb{C}$ conjugacy σ coincides with usual complex conjugation. We use the notation $\bar{\alpha}$ for the complex conjugate of any section α .

If $z = x + iy$ is a local complex coordinate on S as above, then $dz^p d\bar{z}^q$ is a local trivialisation of $\Omega^{[p,q]}S$, and its complex conjugate is $dz^q d\bar{z}^p$.

- (4) In the special case $p = q = 1$ we obtain that $dz d\bar{z} = dx^2 + dy^2$ is real. Further, the local expression of the metric in the local coordinates x, y is $g = \lambda(dx^2 + dy^2)$. So we can consider g as a global section of $\Omega^{[1,1]}S$.

The metric volume form $\omega = \lambda dx \wedge dy$ is the real section of the bundle $\Lambda^2 S \otimes \mathbb{C}$ of complex 2-forms on S . Moreover, under natural isomorphisms $\Omega^{[1,1]}S \xrightarrow{\cong} \Lambda^2 S \otimes \mathbb{C}$ the metric form $g = \lambda(dx^2 + dy^2)$ is mapped onto the volume form $\omega = \lambda dx \wedge dy$. The difference between $\Omega^{[1,1]}S$ and $\Lambda^2 S \otimes \mathbb{C}$ lies in that how these bundles are embedded in $T^*S \otimes T^*S \otimes \mathbb{C}$.

- (5) Let ∇ be the metric covariant derivative on S . Then it can be applied to any section α of any bundle $\Omega^{[p,q]}S$ (or $T^{[p,q]}S$) and the result lies in the product $\Omega^{[p,q]}S \otimes (T^*S)^{\mathbb{C}}$ (or respectively $T^{[p,q]}S \otimes (T^*S)^{\mathbb{C}}$) which is the sum $\Omega^{[p+1,q]}S \oplus \Omega^{[p,q+1]}S$ (respectively $T^{[p+1,q]}S \oplus T^{[p,q+1]}S$). We denote by $\nabla^{(1,0)}$ and $\nabla^{(0,1)}$ the component of $\nabla\alpha$ lying in $\Omega^{[p+1,q]}S$ and $\Omega^{[p,q+1]}S$, respectively. In “dual” notation, $\nabla^{(1,0)}$ and $\nabla^{(0,1)}$ act from $T^{[p,q]}S$ to $T^{[p+1,q]}S$ and $T^{[p,q+1]}S$, respectively.

Explicit calculation show that in coordinates $z = x + iy$ with $g = \lambda(dx^2 + dy^2)$ one has

$$\begin{aligned}\nabla^{(1,0)}(f dz^p d\bar{z}^q) &= \left(\frac{\partial}{\partial z} f - p \frac{\partial \lambda}{\lambda \partial z} f \right) dz^{p+1} d\bar{z}^q \\ \nabla^{(0,1)}(f dz^p d\bar{z}^q) &= \left(\frac{\partial}{\partial \bar{z}} f - q \frac{\partial \lambda}{\lambda \partial \bar{z}} f \right) dz^p d\bar{z}^{q+1}\end{aligned}$$

In particular, $\nabla^{(1,0)}g = \nabla^{(0,1)}g = 0$, $\nabla^{(0,1)}$ coincides with $\frac{\partial}{\partial \bar{z}}$ for sections of holomorphic bundles $T^{[p,0]}S$, $\Omega^{[p,0]}S$, so that such a section α is holomorphic iff $\nabla^{(1,0)}\alpha = 0$.

- (6) The operators $\nabla^{(1,0)}$ and $\nabla^{(0,1)}$ of complex and anti-complex covariant differentiation do not commute. We have

$$(\nabla^{(1,0)}\nabla^{(0,1)} - \nabla^{(0,1)}\nabla^{(1,0)})f dz^p d\bar{z}^q = \frac{q-p}{2} \cdot R \cdot g \cdot f dz^p d\bar{z}^q \quad (2.3)$$

where R is the Gauss curvature of the metric g . The formula is deduced as follows. First, one computes the formula directly for the case $f \equiv 1$. Then using Leibniz rule one shows that $(\nabla^{(1,0)}\nabla^{(0,1)} - \nabla^{(0,1)}\nabla^{(1,0)})$ is not a differential operator but a bundle homomorphism, and hence the formula holds for arbitrary f .

- (7) To simplify notation, we denote the complex and anti-complex covariant differentiations $\nabla^{(1,0)}$ and $\nabla^{(0,1)}$ by $(\cdot)_{;z}$ and $(\cdot)_{;\bar{z}}$, respectively. Iterated differentiations are denoted like $(\cdot)_{;z\bar{z}\bar{z}z}$ and so on, the order of symbols z/\bar{z} coincides with the order of corresponding differentiations. In particular, the formula (2.3) now reads

$$\begin{aligned} (fdz^p d\bar{z}^q)_{;z\bar{z}} - (fdz^p d\bar{z}^q)_{;\bar{z}z} &= \frac{p-q}{2} \cdot R \cdot g \cdot fdz^p d\bar{z}^q; \\ (f\partial_z^p \partial_{\bar{z}}^q)_{;z\bar{z}} - (f\partial_z^p \partial_{\bar{z}}^q)_{;\bar{z}z} &= \frac{q-p}{2} \cdot R \cdot g \cdot f\partial_z^p \partial_{\bar{z}}^q. \end{aligned} \quad (2.4)$$

- (8) We shall use the formula

$$f_{;z}h_{;\bar{z}} - f_{;\bar{z}}h_{;z} = \frac{i}{2}g\{f, h\}_g \quad (2.5)$$

for the Poisson bracket of functions f and h on S with respect to the metric symplectic form $\omega_g = \lambda dx \wedge dy = \frac{i}{2}\lambda dz \wedge d\bar{z}$. Its calculation is as follows:

$$f_{;z}h_{;\bar{z}} - f_{;\bar{z}}h_{;z} = \frac{df \wedge dh}{dz \wedge d\bar{z}} dz d\bar{z} = \frac{i}{2} \frac{df \wedge dh}{\frac{i}{2}\lambda dz \wedge d\bar{z}} \lambda dz d\bar{z} = \frac{i}{2}g\{f, h\}_g.$$

- (9) Sections of bundles $\Omega^{[k,0]}S$, especially holomorphic ones, are called *k-differentials*. For this reason we call sections of bundles $T^{[k,0]}S$ *k-codifferentials*.

Let us now recalculate the Poisson bracket $\{F, H\}$. Recall that we consider the functions F on T^*S which are polynomial in momenta. In local coordinates z, \bar{z}, p, \bar{p} , such a function is a polynomial in p, \bar{p} whose coefficients are smooth functions on S . With every such polynomial $\sum_{i,j=0}^d f_{ij} p^i \bar{p}^j$ we associate a section of the bundle sum $\bigoplus_{i,j=0}^d T^{[i,j]}S$ using the rule $p \mapsto \frac{\partial}{\partial z}, \bar{p} \mapsto \frac{\partial}{\partial \bar{z}}$ and extending it in the obvious way on polynomials. Let us denote this map by $\mathbf{T} : F \mapsto \mathbf{T}(F)$. In particular, $\mathbf{T}(H) = \frac{1}{2}g^{-1}$.

Lemma 2.2. $\mathbf{T}(\{F, H\}) = \mathbf{T}(H) \cdot \nabla(\mathbf{T}(F))$.

Proof. Let us observe that since both sides satisfy the Leibniz rule in F and commute with complex conjugation, it is sufficient to check the formula in the case $F = p$. The rest follows. \square

In view of the lemma, we can replace the functions on T^*S polynomial in momenta by their \mathbf{T} -images which are finite sums of sections of $T^{[i,j]}S$. In our special case first integrals cubic in momenta correspond to sums $\frac{1}{2}(A + \bar{A} + B + \bar{B}) = \Re(A + B)$ where A is a section of $T^{[3,0]}S$ and B is a section of $T^{[2,1]}S$. The equation $\Re(\nabla A + \nabla B) = 0$ takes values in the sum $\bigoplus_{j=-1}^3 T^{[2-j,j]}S$, and homogeneous components of the equation are: $\nabla^{(0,1)}A = 0$, $\nabla^{(1,0)}A + \nabla^{(0,1)}B = 0$, the complex conjugates of these two equations, and $\Re(\nabla^{(1,0)}B) = 0$. As we have seen above, the equation $\nabla^{(0,1)}A = 0$ on a section of $T^{[3,0]}S$ means its holomorphicity.

2.5. Component B and the principle equation. Let us now consider the last equation $\Re(\nabla^{(1,0)}B) = 0$. The tensor B is a section of $T^{[2,1]}S$, so $g^2 \cdot B$ is section of the bundle $\Omega^{[0,1]}S$. Hence $g^2 \cdot B$ has the form $g^2 \cdot B = \beta(x, y) \cdot d\bar{z}$ for some complex function $\beta(x, y) = \beta_1(x, y) + i \cdot \beta_2(x, y)$ with real and imaginary components β_1, β_2 . By property (5) on page 14,

$$g^2 \nabla^{(1,0)}B = \nabla^{(1,0)}(g^2 B) = \frac{\partial \beta}{\partial z} dz d\bar{z}.$$

Since $dz d\bar{z}$ is real, the equation $\Re(\nabla^{(1,0)}B) = 0$ is equivalent to

$$\Re\left(\frac{\partial \beta}{\partial z}\right) = \frac{\partial \beta_1}{\partial x} + \frac{\partial \beta_2}{\partial y} = 0.$$

This equation can be seen as the closedness of the 1-form $-\beta_2 dx + \beta_1 dy$. Hence the equation $\Re(\nabla^{(1,0)}B) = 0$ is equivalent to a local existence of some *real* function $K(x, y)$ such that $\beta_1 = \frac{\partial K}{\partial y}$ and $\beta_2 = -\frac{\partial K}{\partial x}$. The latter two conditions can be rewritten as a single complex equation $\beta = -2i\frac{\partial K}{\partial \bar{z}}$, or equivalently

$$B = -2ig^{-2}K_{;\bar{z}}. \quad (2.6)$$

The latter equation is identical with the equations (2.7), (2.8) in [Du-Ma-To].

Substituting (2.6) in the remaining equation $\nabla^{(1,0)}A + \nabla^{(0,1)}B = 0$, we obtain the following necessary and sufficient condition:

Lemma 2.3. *Given a metric g and a holomorphic 3-codifferential A on a surface S , A is compatible with g if and only if the equation*

$$\mathcal{E}_z : K_{;\bar{z}\bar{z}} = -\frac{i}{2}g^2 A_{;z} \quad (2.7)$$

has a smooth real-valued solution K , and in this case $F = \Re(A + B)$ with $B = -2ig^{-2}K_{;\bar{z}}$ is a cubic integral with the Birkhoff-Kolokoltsov tensor A .

We call (2.7) the *principle equation*.

3. PROOF OF THEOREM 1.1.

3.1. Calculation of further equations. We apply the prolongation-projection method to the equation (2.7) considering the function K as an unknown function and g, A (and their derivatives) as known parameters. The equations obtained by prolongation-projection procedure are called *deduced*. Instead of considering the real and imaginary parts of equations, we shall mostly use complex equations and their complex conjugates. In particular, the complex conjugate to (2.7) is

$$K_{;zz} = \frac{i}{2}g^2 \bar{A}_{;\bar{z}} \quad (3.1)$$

Differentiating (2.7) and (3.1) with respect to z and \bar{z} (first prolongation), we obtain the following 4 equations:

$$K_{;zzz} = \frac{i}{2}g^2 \bar{A}_{;\bar{z}z} \quad K_{;zz\bar{z}} = \frac{i}{2}g^2 \bar{A}_{;\bar{z}\bar{z}} \quad K_{;\bar{z}zz} = -\frac{i}{2}g^2 A_{;zz} \quad K_{;\bar{z}\bar{z}\bar{z}} = -\frac{i}{2}g^2 A_{;z\bar{z}} \quad (3.2)$$

This gives us the expression of all partial derivatives of K of order 3 via lower order derivatives, which are “hidden” due to the covariant form of the equations and do not appear explicitly. In particular, the system $(\mathcal{E}_z, \mathcal{E}_{\bar{z}})$ has finite type.

Differentiating once more (second prolongation), we obtain 6 equations of order 4: 3 as order-2 derivatives of (2.7) and 3 more as derivatives of its complex conjugates:

$$\begin{aligned} K_{;zzzz} &= \frac{i}{2}g^2 \bar{A}_{;\bar{z}zz} & K_{;zzz\bar{z}} &= \frac{i}{2}g^2 \bar{A}_{;\bar{z}\bar{z}\bar{z}} & K_{;zz\bar{z}\bar{z}} &= \frac{i}{2}g^2 \bar{A}_{;\bar{z}\bar{z}\bar{z}} \\ K_{;\bar{z}zzz} &= -\frac{i}{2}g^2 A_{;zzz} & K_{;\bar{z}\bar{z}zz} &= -\frac{i}{2}g^2 A_{;zz\bar{z}} & K_{;\bar{z}\bar{z}\bar{z}\bar{z}} &= -\frac{i}{2}g^2 A_{;z\bar{z}\bar{z}} \end{aligned} \quad (3.3)$$

Since there are 5 partial derivatives of K of order 4, at this prolongation-projection step we can obtain one new equation.

Comparing $\Omega^{[p,q]}$ -types of expressions in (3.3) we see that the desired equation should be the difference of equations $K_{;zz\bar{z}\bar{z}} = \frac{i}{2}g^2 \bar{A}_{;\bar{z}\bar{z}\bar{z}}$ and $K_{;\bar{z}\bar{z}zz} = -\frac{i}{2}g^2 A_{;zz\bar{z}}$:

$$(K_{;zz\bar{z}\bar{z}} - K_{;\bar{z}\bar{z}zz}) - \frac{i}{2}g^2(\bar{A}_{;\bar{z}\bar{z}\bar{z}} + A_{;zz\bar{z}}) \quad (3.4)$$

Computing the difference $K_{;zz\bar{z}\bar{z}} - K_{;\bar{z}\bar{z}zz}$ we use the formula for commutator of covariant derivatives, see property (6) on page 14. This gives

$$\begin{aligned} & K_{;zz\bar{z}\bar{z}} - K_{;\bar{z}\bar{z}zz} = \\ & (K_{;zz\bar{z}\bar{z}} - K_{;z\bar{z}z\bar{z}})_1 + (K_{;z\bar{z}z\bar{z}} - K_{;z\bar{z}\bar{z}z})_2 + (K_{;z\bar{z}\bar{z}z} - K_{;\bar{z}z\bar{z}z})_3 + (K_{;\bar{z}z\bar{z}z} - K_{;\bar{z}\bar{z}zz})_4 = \\ & ((\frac{1}{2}RgK_{;z})_{;\bar{z}})_1 - ((\frac{1}{2}RgK_{;\bar{z}})_{;z})_4 = \frac{g}{2}(K_{;z}R_{;\bar{z}} - K_{;\bar{z}}R_{;z}). \end{aligned}$$

Here we use (2.3) which shows that the differences $(\dots)_2$ and $(\dots)_3$ vanish and gives the formulas above for the differences $(\dots)_1$ and $(\dots)_4$. So using (2.5) we can write down the first deduced equation (new equation in the terminology of §2.3).

$$\{K, R\} = 4\mathfrak{R}(A_{;zzz}) =: D_0 \quad (3.5)$$

where we denote the right hand side by $D_0 =: 4\mathfrak{R}(A_{;zzz}) = 2A_{;zzz} + 2\bar{A}_{;\bar{z}\bar{z}\bar{z}}$.

Remark. We see that the first deduced equation does not contain second derivatives of K . This is not a coincidence but follows from the condition that the higher coefficient in the (2.7) is constant, see [Kr-Ly] for details.

At this point we can solve the problem of existence of cubic integrals with a given Birkhoff-Kolokoltsov tensor A for metrics with constant curvature.

Proposition 3.1. *Let g be a metric metric on a surface S of constant curvature and A a holomorphic 3-codifferential. Then A is compatible with g if and only if A satisfy the PDE $D_0 = 0$.*

Moreover, for a given point $P \in S$ there exists a unique cubic integral of the form $F = \mathfrak{R}(A) + 2H \cdot b^i p_i$ with a vector field $\vec{b} := b^i \partial_{x^i}$ having prescribed values $\vec{b}(P) = \vec{v} \in T_P S$ and $\text{rot } \vec{b}(P) = w \in \mathbb{R}$ at the point P .

Proof. Let (x^1, x^2) be any given coordinate system on S . The formula (5.7) gives the covariant tensor form of the principle equation (2.7). Denote by \mathcal{E}_{ij} the components of this tensor form. Let \mathcal{S} be the system consisting of equations \mathcal{E}_{ij} and their 1st order derivatives $\mathcal{E}_{ij;k}$. Introduce new variables $\mathbf{K}_i = \mathbf{K}_i(x^1, x^2)$, $\mathbf{K}_{ij} = \mathbf{K}_{ij}(x^1, x^2)$, substitute the latter into equations \mathcal{E}_{ij} instead of the corresponding derivatives $K_{;ij}$, and denote the obtained inhomogeneous linear-algebraic equations by $\mathcal{E}_{ij}(\mathbf{K})$. Consider the system $\mathcal{S}(\mathbf{K})$ consisting of equations $\mathcal{E}_{ij}(\mathbf{K})$, their partial derivatives $\mathcal{E}_{ij}(\mathbf{K})_{;k}$, the equations $(\mathbf{K}_i)_{;j} = \mathbf{K}_{ij}$ and the consistency equations $\mathbf{K}_{12} = \mathbf{K}_{21}$ and $(\mathbf{K}_{ij})_{;k} = (\mathbf{K}_{ik})_{;j}$. Then the systems \mathcal{S} and $\mathcal{S}(\mathbf{K})$ are equivalent: the inverse substitution of $K_{;i}, K_{;ij}$ instead of $\mathbf{K}_i, \mathbf{K}_{ij}$ transforms $\mathcal{S}(\mathbf{K})$ into \mathcal{S} (turning consistency equations into identities). Moreover, both system are involutive simultaneously.

Now, since the system $\mathcal{S}(\mathbf{K})$ consists of 1st order linear PDEs and is involutive, it admits a local solution by the classical Frobenius theorem (see eg. [Iv-La], § 1.9). Moreover, for every $P \in S$ and every values $\mathbf{K}_i(P), \mathbf{K}_{ij}(P)$ satisfying algebraic equations $\mathcal{E}_{ij}(\mathbf{K})$ at the point P there exists a local solution $(\mathbf{K}_i(x), \mathbf{K}_{ij}(x))$. For this \mathbf{K} -solution there exists a function $K(x)$, unique up to adding a constant, which satisfies the relations $K_{;i} = \mathbf{K}_i, K_{;ij} = \mathbf{K}_{ij}$.

To calculate the number of parameters, we use linear-algebraic equations $\mathcal{E}_{ij}(\mathbf{K})$ and express the functions $\mathbf{K}_{12} = \mathbf{K}_{21}$ and \mathbf{K}_{22} via $\mathbf{K}_1, \mathbf{K}_2$, and \mathbf{K}_{11} . Substituting these formulas in the remaining PDEs, we reduce the number of unknown functions to three. Let $\mathcal{S}'(\mathbf{K})$ be the obtained system. Notice that some of the substitutions above mean the application of prolongation-projection procedure, for example, substitution of the formulas $\partial_i \mathbf{K}_{jk} = \mathcal{F}_{ijk}$ in equation $(\mathbf{K}_{ij})_{;k} = (\mathbf{K}_{ik})_{;j}$. However, the involutivity of $\mathcal{S}(\mathbf{K})$ implies that every equation

in $\mathcal{S}'(\mathbf{K})$ is still a *linear-algebraic* consequence of the system $\mathcal{S}(\mathbf{K})$. In turn, this implies the involutivity of the system $\mathcal{S}'(\mathbf{K})$: Indeed, every differential consequence of $\mathcal{S}'(\mathbf{K})$ is a differential consequence of $\mathcal{S}(\mathbf{K})$, and hence is a linear-algebraic consequence.

By the construction, the system $\mathcal{S}(\mathbf{K})$ is linear-algebraic equivalent to a system containing the equations of the form $\partial_i \mathbf{K}_j = \mathcal{F}_{ij}$, $\partial_i \mathbf{K}_{jk} = \mathcal{F}_{ijk}$ where \mathcal{F}_{ij} , \mathcal{F}_{ijk} are linear inhomogeneous expressions in \mathbf{K}_i , \mathbf{K}_{ij} . It follows that the system $\mathcal{S}'(\mathbf{K})$ is also linear-algebraic equivalent to a system *consisting of* the equations of the form $\partial_i \mathbf{K}_j = \mathcal{F}'_{ij}$, $\partial_i \mathbf{K}_{11} = \mathcal{F}'_{i11}$ where \mathcal{F}'_{ij} , \mathcal{F}'_{i11} are as above linear inhomogeneous in \mathbf{K}_i , \mathbf{K}_{11} , and we may assume that the system $\mathcal{S}'(\mathbf{K})$ is of this form.

Now the generic solution of the system $\mathcal{S}'(\mathbf{K})$ is constructed as follows: Given a point P and values $\mathbf{K}_1(P)$, $\mathbf{K}_{12}(P)$, $\mathbf{K}_{11}(P)$ at P , we fix local coordinates $x := x^1$ and $y := x^2$, integrate the ODE system ($\partial_x \mathbf{K}_i = \mathcal{F}_{1i}$, $\partial_x \mathbf{K}_{11} = \mathcal{F}_{111}$) along some interval on the x -axis, and then integrate the ODE systems ($\partial_y \mathbf{K}_i = \mathcal{F}_{2i}$, $\partial_y \mathbf{K}_{11} = \mathcal{F}_{211}$) along every interval parallel to y -axis.

It remains to show that the prescribed values as in the assertion of the proposition can be used to parametrised a general solution. Indeed, as such parameters we can use the values of the 1st order derivatives of K and the Laplacian $\Delta_g K$ (instead of $K_{;11}$ at a given point $P \in S$. Now recall that by § 2.5, b^i and K are related as $K_{;i} = \omega_{ij} b^j$ so that \vec{b} is the Hamiltonian vector field on S of the (Hamiltonian) function K (with respect to the form ω_g). Consequently, \vec{b} is the skew-gradient of K and the Laplacian $\Delta_g K$ is the rotor $\text{rot } \vec{b}$. \square

3.2. Proof of Theorem 1.1. Let us now turn to the (most interesting) general case when R is non-constant. Write the equation (3.5) as $K_{;z} R_{;\bar{z}} - K_{;\bar{z}} R_{;z} = \frac{i}{2} g D_0$. Differentiating it in z and \bar{z} we obtain two more equations of order 2 in K :

$$\begin{aligned} K_{;zz} R_{;\bar{z}} + K_{;z} R_{;\bar{z}z} - K_{;\bar{z}z} R_{;z} - K_{;\bar{z}} R_{;zz} &= \frac{i}{2} g \cdot (D_0)_z \\ K_{;z\bar{z}} R_{;\bar{z}} + K_{;z} R_{;\bar{z}\bar{z}} - K_{;\bar{z}\bar{z}} R_{;z} - K_{;\bar{z}} R_{;z\bar{z}} &= \frac{i}{2} g \cdot (D_0)_{\bar{z}} \end{aligned} \quad (3.6)$$

Together with (2.7) and its conjugate we now have 4 equations of second order on K . So we expect to obtain at this step one new equation of order 1 on K . It is obtained as follows:

Add equations (3.6) with coefficients $R_{;\bar{z}}$ and $R_{;z}$:

$$\begin{aligned} K_{;zz} R_{;\bar{z}} R_{;\bar{z}} + K_{;z} R_{;\bar{z}z} R_{;\bar{z}} - K_{;\bar{z}z} R_{;z} R_{;\bar{z}} + \\ K_{;z} R_{;\bar{z}\bar{z}} R_{;z} - K_{;\bar{z}\bar{z}} R_{;z} R_{;z} - K_{;\bar{z}} R_{;z\bar{z}} R_{;z} &= \frac{i}{2} g \cdot (D_0)_{;z} R_{;\bar{z}} + \frac{i}{2} g \cdot (D_0)_{;\bar{z}} R_{;z} \end{aligned}$$

Rearranging it, we obtain

$$K_{;z} (R_{;z} R_{;\bar{z}})_{;\bar{z}} - K_{;\bar{z}} (R_{;z} R_{;\bar{z}})_{;z} = \frac{i}{2} g \cdot (D_0)_{;z} R_{;\bar{z}} + \frac{i}{2} g \cdot (D_0)_{;\bar{z}} R_{;z} + K_{;\bar{z}\bar{z}} R_{;z} R_{;z} - K_{;zz} R_{;\bar{z}} R_{;\bar{z}}.$$

The expression $R_{;z} R_{;\bar{z}}$ equals $\frac{1}{4} g |\nabla R|^2$. Let us denote $\frac{1}{2} |\nabla R|^2$ by φ_1 . So the left hand side of the latter equation is $\frac{i}{4} g^2 \{K, \varphi_1\}$. On the right hand side we have $\frac{i}{2} g ((D_0)_{;z} R_{;\bar{z}} + (D_0)_{;\bar{z}} R_{;z})$ which equals $\frac{i}{4} g^2 \langle \nabla D_0, \nabla R \rangle$. In the last two terms we use (2.7). So finally we obtain the equation

$$\{K, \varphi_1\} = \langle \nabla D_0, \nabla R \rangle - 4\Re(A_{;z} \cdot (R_{;z})^2) =: D_1, \quad (3.7)$$

in which we denoted the right hand side by D_1 .

From this points we can produce many equations, each of the form $\{K, \varphi\} = D$ where φ and D are real functions which are certain differential expressions involving the metric g and the tensor $A_{;z}$. In fact, we have two such ‘‘dummy’’ procedures:

The first one is the formal repetition of the deduction (3.5) \implies (3.7) and uses the fact that the explicit expressions for R and D_0 were not involved. So from any equation $\{K, \varphi\} = D$ we can obtain a new equation $\{K, \varphi'\} = D'$ with $\varphi' := \frac{1}{2}|\nabla\varphi|^2$ and $D' := \langle \nabla D, \nabla\varphi \rangle - 4\Re(A_{;z}\varphi_{;z}^2)$.

The second procedure is based on the Jacobi identity for Poisson brackets: starting with any two equations of the form $\{K, \varphi'\} = D'$ and $\{K, \varphi''\} = D''$, we obtain

$$\{K, \{\varphi', \varphi''\}\} = \{\{K, \varphi'\}, \varphi''\} - \{\{K, \varphi''\}, \varphi'\} = \{D', \varphi''\} - \{D'', \varphi'\} \quad (3.8)$$

and we can set $\varphi''' := \{\varphi', \varphi''\}$ and $D''' := \{D', \varphi''\} - \{D'', \varphi'\}$.

Let us notice that both procedures are application of prolongation-projection Method: In the first case we differentiate the equation $\{K, \varphi\} = D$ in z and \bar{z} , add the equation (2.7) and its complex conjugate, and make linear-algebraic manipulations on the system of 4 equations excluding second order derivatives of K . In the second case we differentiate both equations $\{K, \varphi'\} = D'$ and $\{K, \varphi''\} = D''$ and then make the same linear-algebraic manipulations on the system of 4 equations.

To unify the notation, we set

$$\begin{aligned} \varphi_0 &:= R, & \varphi_1 &:= \frac{1}{2}|\nabla\varphi_0|^2, & \varphi_2 &:= \{\varphi_0, \varphi_1\}, & \varphi_3 &:= \frac{1}{2}|\nabla\varphi_1|^2, \\ D_0 &:= 4\Re(A_{;zzz}), & D_1 &:= \langle \nabla D_0, \nabla\varphi_0 \rangle - 4\Re(A_{;z} \cdot ((\varphi_0)_{;z})^2), & & & & \\ D_2 &:= \{D_0, \varphi_1\} - \{D_1, \varphi_0\}, & D_3 &:= \langle \nabla D_1, \nabla\varphi_1 \rangle - 4\Re(A_{;z} \cdot ((\varphi_1)_{;z})^2). \end{aligned} \quad (3.9)$$

Thus the equation $\{K, \varphi_2\} = D_2$ is obtained from $\{K, \varphi_0\} = D_0$ and $\{K, \varphi_1\} = D_1$ using Jacobi identity, and the equation $\{K, \varphi_3\} = D_3$ from $\{K, \varphi_1\} = D_1$ using (3.5) and its complex conjugate.

It appears that the system consisting of 4 equations $\mathcal{E}_i: \{K, \varphi_i\} = D_i$, $i = 0, \dots, 3$ is involutive and differentially equivalent to the original equation \mathcal{E}_z . We state a more general property which will be used also in the proof of Theorem 1.2.

Lemma 3.1. (a) *Let the coefficients of the equations $\mathcal{E}' = \{K, \varphi'\} - D'$, $\mathcal{E}'' = \{K, \varphi''\} - D''$, and $\mathcal{E}''' = \{K, \varphi'''\} - D'''$ are related as $\varphi''' := \{\varphi', \varphi''\}$ and $D''' := \{D', \varphi''\} - \{D'', \varphi'\}$. Then the equation \mathcal{E}''' is a linear-algebraic consequence of the equations $\mathcal{E}'_{;x}, \mathcal{E}'_{;y}, \mathcal{E}''_{;x}, \mathcal{E}''_{;y}$.*

(b) *Let the coefficients of the equations $\mathcal{E}' = \{K, \varphi\} - D$ and $\mathcal{E}^* = \{K, \varphi^*\} - D^*$ are related as $\varphi^* := \frac{1}{2}|\nabla\varphi|^2$ and $D^* := \langle \nabla D', \nabla\varphi' \rangle - 4\Re(A_{;z}(\varphi'_{;z})^2)$. Then the equation \mathcal{E}^* is a linear-algebraic consequence of the equations $\mathcal{E}'_{;x}, \mathcal{E}'_{;y}$, and the equations*

(I) \mathcal{E}_z and its conjugate $\mathcal{E}_{\bar{z}}$.

(c) *Under hypotheses of parts (a) and (b) assume additionally that $\{\varphi', \varphi''\} = \varphi'''$ is non-vanishing at a generic point. Let also the equation $\mathcal{E}^\dagger = \{K, \varphi^\dagger\} - D^\dagger$ be defined by $\varphi^\dagger := \frac{1}{2}|\nabla\varphi''|^2$ and $D^\dagger := \langle \nabla D'', \nabla\varphi'' \rangle - 4\Re(A_{;z}(\varphi''_{;z})^2)$. Then the equations (I) are linear-algebraic consequences of the following set of equations:*

(II) equations $\mathcal{E}''', \mathcal{E}^*, \mathcal{E}^\dagger$;

(III) the first order derivatives $\mathcal{E}'_{;x}, \mathcal{E}'_{;y}, \mathcal{E}''_{;x}, \mathcal{E}''_{;y}$.

Recall that ‘‘linear-algebraic consequence’’ means that new equations appear as linear combinations of old ones with coefficients which are rational functions of coefficients of the old equations.

Proof. Clearly, parts (a) and (b) are simply a restating of the “dummy” procedures introduced above.

(c) Rewrite the equation \mathcal{E}' and \mathcal{E}'' in the form $K_z\varphi'_{;z} - K_{\bar{z}}\varphi'_{;z} = 2igD'$ and $K_z\varphi''_{;z} - K_{\bar{z}}\varphi''_{;z} = 2igD''$. Then the determinant of the linear system is $\varphi'_{;z}\varphi''_{;z} - \varphi''_{;z}\varphi'_{;z} = -2ig \cdot \{\varphi', \varphi''\}$, and the solution is given by the formulas $K_i = \mathcal{F}_i$ with $\mathcal{F}_i := \frac{1}{\{\varphi', \varphi''\}} \cdot \det \begin{pmatrix} D' & -\varphi'_{;x^i} \\ D'' & -\varphi''_{;x^i} \end{pmatrix}$. Differentiating the formulas, we conclude that there exists a *unique* solutions of the equations (III), considered as a system of linear equations on $K_{;x^i x^j}$, and this solution is given by $K_{;x^i x^j} = (\mathcal{F}_i)_{;x^j}$. In particular, $(\mathcal{F}_1)_{;x^2} = (\mathcal{F}_2)_{;x^1}$. Notice that each $(\mathcal{F}_i)_{;x^j}$ has the form $\frac{\tilde{\mathcal{F}}_{ij}}{\{\varphi', \varphi''\}^2}$ for some PDOs $\tilde{\mathcal{F}}_{ij}$ which are *polynomial* in $\varphi', \varphi'', D', D''$ and their derivatives. This shows the fact that the determinant of the matrix of leading coefficients of the system (III) equals $\{\varphi', \varphi''\}^2$ (up to sign). Indeed, the solutions $K_{;x^i x^j} = (\mathcal{F}_i)_{;x^j}$ can be obtained by means of linear-algebraic manipulations with equations (III).

The key point in the proof is that we can obtain the equations (II) substituting the expressions $K_{;x^i x^j} = (\mathcal{F}_i)_{;x^j}$ in the equations (I). This is also linear-algebraic manipulations with equations. The assertion (c) claims that this operation is invertible, which is true. \square

Define the expressions \mathcal{G}_i using the following Jacobi-like expressions:

$$\begin{aligned} \mathcal{G}_0 &= \{\varphi_1, \varphi_2\} \cdot D_3 + \{\varphi_2, \varphi_3\} \cdot D_1 + \{\varphi_3, \varphi_1\} \cdot D_2, \\ \mathcal{G}_1 &= \{\varphi_0, \varphi_2\} \cdot D_3 + \{\varphi_2, \varphi_3\} \cdot D_0 + \{\varphi_3, \varphi_0\} \cdot D_2, \\ \mathcal{G}_2 &= \{\varphi_0, \varphi_1\} \cdot D_3 + \{\varphi_1, \varphi_3\} \cdot D_0 + \{\varphi_3, \varphi_0\} \cdot D_1, \\ \mathcal{G}_3 &= \{\varphi_0, \varphi_1\} \cdot D_2 + \{\varphi_1, \varphi_2\} \cdot D_0 + \{\varphi_2, \varphi_0\} \cdot D_1. \end{aligned}$$

Up to normalisation, the expressions \mathcal{G}_i are the (3×3) -minors of the extended matrix of the coefficients of the equations $\mathcal{E}_0, \dots, \mathcal{E}_3$ with i -th row excluded. In particular,

$$\mathcal{G}_2 := \frac{i}{2g} \det \begin{pmatrix} (\varphi_0)_{;z} & (\varphi_0)_{;\bar{z}} & D_0 \\ (\varphi_1)_{;z} & (\varphi_1)_{;\bar{z}} & D_1 \\ (\varphi_3)_{;z} & (\varphi_3)_{;\bar{z}} & D_3 \end{pmatrix} \quad \text{and} \quad \mathcal{G}_3 := \frac{i}{2g} \det \begin{pmatrix} (\varphi_0)_{;z} & (\varphi_0)_{;\bar{z}} & D_0 \\ (\varphi_1)_{;z} & (\varphi_1)_{;\bar{z}} & D_1 \\ (\varphi_2)_{;z} & (\varphi_2)_{;\bar{z}} & D_2 \end{pmatrix}.$$

Every expression \mathcal{G}_i is a PDE on the metric g and the 3-codifferential A .

Corollary 3.1. *Assume that φ_2 is non-vanishing. Then the system $\mathcal{S} := (\mathcal{E}_0, \mathcal{E}_1)$ is involutive if and only if the integrability condition $\mathcal{G}_3 = 0$ is fulfilled. A solution K of this system satisfies the equation $K_{;\bar{z}\bar{z}} = -\frac{i}{2}g^2 A_{;z}$ if and only if the integrability condition $\mathcal{G}_2 = 0$ is fulfilled.*

At this point we can give the proof of Theorem 1.1.

Proposition 3.2. *Let $g = \lambda(x, y)(dx^2 + dy^2)$ be a metric and A a holomorphic 3-codifferential such that the differential expression $\varphi_2 = \{\varphi_0, \varphi_1\}$ is non-vanishing.*

Then g admits a cubic integral of the form $F = \Re(A + bp^2\bar{p})$ if and only if λ satisfies the covariant PDEs $\mathcal{G}_3, \mathcal{G}_2$. Moreover, in this case the complex valued function $b = b_1 + ib_2$ is given by the formulas $b_1 = \lambda^{-2}\mathcal{K}_2, b_2 = -\lambda^{-2}\mathcal{K}_1$ with

$$\mathcal{K}_i := \frac{1}{\{\varphi_0, \varphi_1\}} \cdot \det \begin{pmatrix} D_0 & -(\varphi_0)_{;x^i} \\ D_1 & -(\varphi_1)_{;x^i} \end{pmatrix}. \quad (3.10)$$

Proof. By Lemma 2.3 from §2.5, the existence of a cubic integral F of the form above is equivalent to the existence of a solution K of the principle equation $K_{;\bar{z}\bar{z}} = -\frac{i}{2}g^2A_{;z}$, and in this case $F = \Re(A + B)$ with $B := bp^2\bar{p} = -2ig^{-2}K_{;\bar{z}}$ is the desired cubic integral.

As we have shown above, equations $\mathcal{E}_0, \mathcal{E}_1$ are differential consequences of the real and imaginary parts of \mathcal{E}_z . So by Corollary 3.1, equations $\mathcal{G}_2 = 0, \mathcal{G}_3 = 0$ are necessary and sufficient conditions for solvability of \mathcal{E}_z . The formulas (3.10) give simply the solution of a linear system $(\mathcal{E}_0, \mathcal{E}_1)$. \square

Let us observe that under condition of non-vanishing of all brackets $\{\varphi_i, \varphi_j\}$ any two of the equations $\mathcal{G}_0, \dots, \mathcal{G}_3$ are linear-algebraic consequence of remaining two.

4. METRICS WITH $\{R, |\nabla R|_g^2\}_g \equiv 0$ AND PROOF OF THEOREM 1.2.

In this section we consider the degenerate case when the function φ_2 introduced in (1.6) vanishes identically. This will be a general assumption in this section unless the opposite is stated explicitly. In this case we additionally suppose that the Gauss curvature $R = \varphi_0$ is non-degenerate, ie., the gradient of R is non-vanishing.

4.1. Proof of Theorem 1.2. Recall that the condition $\varphi_2 \equiv 0$ means that the Gauss curvature R and the square of its gradient $|\nabla R|^2$ are functionally dependent, $\frac{1}{2}|\nabla R|^2 = f(R)$ for some function $f(r)$ of one variable. Notice that the function $\varphi_3 = \frac{1}{2}|\nabla\varphi_1|^2$ is also functionally dependent, since

$$\frac{1}{2}|\nabla\varphi_1|^2 = \frac{1}{2}|\nabla f(R)|^2 = (f'(R))^2 \frac{1}{2}|\nabla R|^2 = (f'(R))^2 f(R).$$

As in the non-degenerate case, we want to apply the prolongation-projection method. Our starting system of equations is again $\mathcal{S}_0 := \{\mathcal{E}_z, \mathcal{E}_{\bar{z}}\}$. Let us analyse the ‘‘forking point’’ in the procedure. Since we assume non-vanishing of ∇R , there will be no divergence in results until Corollary 3.1. At this point the degenerate case $\varphi_2 \equiv 0$ differs from the non-degenerate one in two aspects.

The first one is that we must replace \mathcal{E}_1 by another equation. We shall find such an equation in a moment. The second aspect is that in the case $\varphi_2 \equiv 0$ the result of the projection procedure is slightly different as before. Namely, now the system $(\mathcal{E}_0, \mathcal{E}_1)$ is degenerate at every point since the left hand sides of the equations, $\{K, \varphi_0\}$ and $\{K, \varphi_1\}$, are Lie derivatives of K along proportional vector fields. Since the equation \mathcal{E}_1 still must be satisfied, we obtain a new differential condition on g and A : the right hand sides of $\mathcal{E}_0, \mathcal{E}_1$ must be proportional with the same coefficient as the left ones. Clearly, this condition is simply the application of the projection procedure to the system $(\mathcal{E}_0, \mathcal{E}_1)$.

The resulting equation reads:

$$\mathcal{D}_z := (\varphi_0)_{;z}D_1 - (\varphi_1)_{;z}D_0 = 0. \quad (4.1)$$

Notice that even if this is formally a complex-valued equation and so two real ones, the real and imaginary parts are equivalent under the condition $\varphi_2 \equiv 0$. This means that the system of conditions $\{\varphi_2, \mathcal{D}_z\}$ contains only two independent conditions.

In a more explicit form the equation \mathcal{D}_z reads:

$$\mathcal{D}_z = R_{;z} \cdot \langle \nabla D_0, \nabla R \rangle - R_{;z} \cdot 4 \cdot \Re(A_{;z} \cdot (R_{;z})^2) - D_0 \cdot (\frac{1}{2}|\nabla R|^2)_{;z}.$$

Another possible form for this condition is the differential 1-form

$$\det \begin{pmatrix} d\varphi_0 & D_0 \\ d\varphi_1 & D_1 \end{pmatrix} = \mathcal{D}_z dz + \mathcal{D}_{\bar{z}} d\bar{z} = \mathcal{D}_x dx + \mathcal{D}_y dy,$$

where $\mathcal{D}_{\bar{z}}, \mathcal{D}_x, \mathcal{D}_y$ are defined in the obvious way. One can use any of the equation $\mathcal{D}_x, \mathcal{D}_y$ instead of \mathcal{D}_z provided the corresponding derivative $\frac{\partial \varphi_0}{\partial x^i}$ is non-vanishing.

Now we seek for the substitute for the equation \mathcal{E}_1 . Since our previous step was the projection, the next one is the prolongation of the equations. Let \mathcal{S} be the system of equations obtained so far. Those are $\mathcal{E}_z, \mathcal{E}_{\bar{z}}$ and their derivatives up to order 2, the equation \mathcal{E}_0 and its derivative $(\mathcal{E}_0)_{;x}$, and the equations φ_2 and \mathcal{D}_z .

As the next step we are going to add to \mathcal{S} the second order derivatives $(\mathcal{E}_0)_{;x^i x^j}$. Here we make the following easy observation. It follows from Lemma 3.1 and the above consideration that the equations $(\mathcal{E}_0)_{;y}$ is a linear-algebraic combination of the equations from \mathcal{S} of order ≤ 2 in K and the condition \mathcal{D}_z . Similarly, $(\mathcal{E}_0)_{;x}$ can be obtained as a linear algebraic combination of the condition \mathcal{D}_z and the equations from \mathcal{S} of order ≤ 2 in K , in which $(\mathcal{E}_0)_{;x}$ is replaced by $(\mathcal{E}_0)_{;y}$. Thus replacing $(\mathcal{E}_0)_{;x}$ by $(\mathcal{E}_0)_{;y}$ we obtain a system equivalent to \mathcal{S} provided $\mathcal{D}_z \equiv 0$.

Consequently, under condition $\mathcal{D}_z \equiv 0$ the derivative $(\mathcal{E}_0)_{;yy}$ is a linear algebraic combination of the same equations and its derivatives in y . In particular, $(\mathcal{E}_0)_{;yyy}$ is a linear algebraic combination of equations from \mathcal{S} and the equation $(\mathcal{E}_0)_{;xy}$. Interchanging x and y we conclude that $(\mathcal{E}_0)_{;xx}$ is also a linear algebraic combination of equations from \mathcal{S} and the equation $(\mathcal{E}_0)_{;xy}$. In fact, each of the equations $(\mathcal{E}_0)_{;x^i x^j}$ is linear-algebraic equivalent to each other modulo the system \mathcal{S} and the condition \mathcal{D}_z . Thus adding to \mathcal{S} an arbitrary single derivative $(\mathcal{E}_0)_{;x^i x^j}$ instead all three we obtain an equivalent new system. As such a derivative we choose $(\mathcal{E}_0)_{;xy} = (\mathcal{E}_0)_{;z\bar{z}}$.

Now we make explicit calculation. The derivation $(\mathcal{E}_0)_{;z\bar{z}}$ gives

$$\begin{aligned} & K_{;zz\bar{z}}R_{;\bar{z}} + K_{;z\bar{z}z}R_{;z\bar{z}} + K_{;z\bar{z}\bar{z}}R_{;z\bar{z}z} + K_{;z\bar{z}}R_{;z\bar{z}z\bar{z}} \\ & - K_{;z\bar{z}\bar{z}z}R_{;z} - K_{;z\bar{z}z}R_{;z\bar{z}z} - K_{;z\bar{z}\bar{z}z}R_{;z\bar{z}z} - K_{;z\bar{z}}R_{;z\bar{z}z\bar{z}} = 2ig \cdot (D_0)_{;z\bar{z}}. \end{aligned} \quad (4.2)$$

Simplifying the obtained equation we apply the following relations: the rule (6), the homomorphism equations $A_{;\bar{z}} = 0, \bar{A}_{;z} = 0$, and the substitutions (2.7), (3.2). Let us notice that the latter case we exclude higher order derivatives of K and so apply the projection procedure. Calculating, we obtain $K_{;zz\bar{z}} = \frac{1}{2}g^2\bar{A}_{;\bar{z}\bar{z}}$ and

$$K_{;z\bar{z}\bar{z}} = K_{;z\bar{z}z} + (K_{;z\bar{z}\bar{z}z} - K_{;z\bar{z}z\bar{z}}) = -\frac{1}{2}g^2A_{;zz} - \frac{1}{2}gRK_{;\bar{z}}.$$

Besides we also use the relations

$$R_{;z\bar{z}} = R_{;\bar{z}z} = \frac{1}{4}g\Delta R \quad \text{and} \quad R_{;zz\bar{z}} = R_{;z\bar{z}z} + \frac{1}{2}gRR_{;z} = \frac{1}{4}g\Delta R_{;z} + \frac{1}{2}gRR_{;z}$$

where $\Delta = \Delta_g$ is the metric Laplace operator. Substitution of these relations yields

$$\begin{aligned} & \frac{1}{2}g^2\bar{A}_{;\bar{z}\bar{z}}R_{;\bar{z}} + \frac{1}{2}g^2\bar{A}_{;\bar{z}}R_{;z\bar{z}} + \frac{1}{4}K_{;z}g\Delta R_{;\bar{z}} + \\ & (\frac{1}{2}g^2A_{;zz} + \frac{1}{2}gRK_{;\bar{z}})R_{;z} + \frac{1}{2}g^2A_{;z}R_{;z\bar{z}} - K_{;\bar{z}}(\frac{1}{4}g\Delta R_{;z} + \frac{1}{2}gRR_{;z}) = \frac{1}{2}g^2 \cdot \Delta D_0. \end{aligned}$$

Rearranging the terms and dividing by $\frac{1}{4}g$, we obtain

$$K_{;z}\Delta R_{;\bar{z}} - K_{;\bar{z}}\Delta R_{;z} = 2ig\Delta D_0 - 2ig(A_{;zz}R_{;z} + A_{;z}R_{;z\bar{z}} + \bar{A}_{;\bar{z}\bar{z}}R_{;\bar{z}} + \bar{A}_{;\bar{z}\bar{z}z}R_{;\bar{z}}),$$

which finally yields

$$\{K, \Delta R\} = \Delta D_0 - 2\Re((A_{;z}R_{;z})_{;z}).$$

We denote this equation by \mathcal{E}_1^* and set

$$\varphi_1^* := \Delta R \quad D_1^* := \Delta D_0 - 2\Re((A_{;z}R_{;z})_{;z}). \quad (4.3)$$

The equation \mathcal{E}_1^* has the already familiar form

$$\{K, \varphi_1^*\} = D_1^*, \quad (4.4)$$

and so we can apply Lemma 3.1. This gives us two more equations

$$\mathcal{E}_2^* := \{K, \varphi_2^*\} - D_2^* \quad \mathcal{E}_3^* := \{K, \varphi_3^*\} - D_3^* \quad (4.5)$$

in which

$$\varphi_2^* := \{\varphi_0, \varphi_1^*\} \quad D_2^* := \{D_0, \varphi_1^*\} - \{D_1^*, \varphi_0\} \quad (4.6)$$

$$\varphi_3^* := \frac{1}{2}|\nabla\varphi_1^*|^2 \quad D_3^* := \langle \nabla D_1^*, \nabla\varphi_1^* \rangle - 4\Re(A_{;z} \cdot ((\varphi_1^*)_{;z})^2). \quad (4.7)$$

We also define the differential expressions

$$\mathcal{K}_i^* := \frac{1}{\{\varphi_0, \varphi_1^*\}} \cdot \det \begin{pmatrix} D_0 & -(\varphi_0)_{;x^i} \\ D_1^* & -(\varphi_1^*)_{;x^i} \end{pmatrix}. \quad (4.8)$$

Proposition 4.1. *Let $g = \lambda(x, y)(dx^2 + dy^2)$ be a metric and A a holomorphic 3-codifferential. Assume that g satisfies the differential condition $\varphi_2 \equiv 0$ and that $\varphi_2^* = \{R, \Delta_g R\}_g$ is non-vanishing.*

Then g admits a cubic integral of the form $F = \Re(A \cdot p^3 + b \cdot p^2 \bar{p})$ with the given tensor A if and only if λ and A satisfy the covariant PDEs $\mathcal{G}_3^, \mathcal{G}_2^*$, and \mathcal{D} . Moreover, the component $b = b_1 + ib_2$ is given by the formulas $b_1 = \lambda^{-2}\mathcal{K}_2^*, b_2 = -\lambda^{-2}\mathcal{K}_1^*$.*

Proof. As above, the existence of such a cubic integral F is equivalent to solvability of the equation \mathcal{E}_z : $K_{;z\bar{z}} = -\frac{1}{2}g^2 A_{;z}$. Since $\mathcal{D}, \mathcal{G}_2^*, \mathcal{G}_3^*$ are differential consequences of \mathcal{E}_z , they are necessary conditions.

Vice versa, the system $(\mathcal{E}_0, \mathcal{E}_1^*)$ is solvable if and only if the integrability condition $\mathcal{G}_3^* \equiv 0$ is fulfilled, and then the solution is given by the formulas (4.8). Further, by Lemma 3.1, (c), the equation \mathcal{E}_z is a linear algebraic consequence of the conditions of the 1st order derivatives $(\mathcal{E}_0)_{;i}, (\mathcal{E}_1^*)_{;i}$ and the equations $\mathcal{E}_1, \mathcal{E}_2^*, \mathcal{E}_3^*$. As we have shown, in the presence of $\mathcal{E}_0, \mathcal{E}_1^*$ those three equations are equivalent to $\mathcal{D}, \mathcal{G}_3^*, \mathcal{G}_2^*$, respectively. Thus under the conditions $\mathcal{D} = 0, \mathcal{G}_2^* = 0, \mathcal{G}_3^* = 0$ the solution K of the system $(\mathcal{E}_0, \mathcal{E}_1^*)$ solves also the equation $K_{;z\bar{z}} = -\frac{1}{2}g^2 A_{;z}$. The proposition follows. \square

4.2. Metrics admitting a Killing vector. In previous paragraphs we considered the cases when one of the differential expressions $\varphi_2 = \{R, \frac{1}{2}|\nabla R|_g^2\}_g$ or $\varphi_2^* = \{R, \Delta_g R\}_g$ is non-zero. Here we treat the problem of detecting of cubic integrals in the case when both φ_2 and φ_2^* vanish. Recall that by Bonnet-Darboux-Eisenhart theorem (see § 1.5) this means that the metric g admits a Killing vector field $L^i \frac{\partial}{\partial x^i}$, and then $L = L^i p_i$ is a non-trivial linear integral.

We maintain the notation introduced above. Define the expressions $\mathcal{D}_x^*, \mathcal{D}_y^*, \mathcal{D}_z^*, \mathcal{D}_{\bar{z}}^*$ from the relations

$$\det \begin{pmatrix} d\varphi_0 & D_0 \\ d\varphi_1^* & D_1^* \end{pmatrix} = \mathcal{D}_z^* dz + \mathcal{D}_{\bar{z}}^* d\bar{z} = \mathcal{D}_x^* dx + \mathcal{D}_y^* dy.$$

In particular,

$$\mathcal{D}_x^* := (\varphi_0)_{;x} D_1^* - (\varphi_1^*)_{;x} D_0 = R_{;x} \cdot (\Delta D_0 - 2\Re((A_{;z} R_{;z})_{;z})) - \Delta R_{;x} \cdot D_0 \quad (4.9)$$

and similarly for $\mathcal{D}_y^*, \mathcal{D}_z^*, \mathcal{D}_{\bar{z}}^*$.

Proposition 4.2. *Let g be a Riemannian metric on a surface S with the curvature R and A a holomorphic 3-codifferential. Assume that $R_{;x}$ is non-vanishing and that the expressions $\varphi_2 = \{R, \frac{1}{2}|\nabla R|_g^2\}_g$ and $\varphi_2^* = \{R, \Delta_g R\}_g$ vanish. Then A is compatible with g if and only if it satisfies the equations $\mathcal{D}_x = 0$ and $\mathcal{D}_x^* = 0$.*

Proof. Applying the prolongation-projection method, we repeat the proof of Proposition 4.1 until computation of the equation \mathcal{E}_1^* . At this point the hypothesis $\varphi_2^* \neq 0$ of the part (c) of Lemma 3.1 is not satisfied. The situation here is similar to that in Section 4 where the hypothesis $\varphi_2 \neq 0$ was not fulfilled.

At this step prolongation-projection method produces the equation $\mathcal{D}_x^* = 0$, which is an analogue of the equation $\mathcal{D}_x = 0$ in the case $\varphi_2^* \equiv 0$. Indeed, the condition $\varphi_2^* \equiv 0$ means the degeneration of the matrix of coefficients of the equations $\mathcal{E}_0, \mathcal{E}_1^*$ whereas the pair of equations $(\varphi_2^*, \mathcal{D}_x^*)$ means the degeneration of the *extended* matrix of coefficients of $\mathcal{E}_0, \mathcal{E}_1^*$. In particular, under the hypotheses $R_{;x} \neq 0$ and $\varphi_2^* = 0$ the equations $\mathcal{E}_1^* = 0$ and $\mathcal{D}_x^* = 0$ are equivalent.

Since $\mathcal{D}_x, \mathcal{D}_x^*$ are differential consequences of the equations $\mathcal{E}_z, \mathcal{E}_{\bar{z}}$, their vanishing is a necessary condition for solvability of \mathcal{E}_z .

Vice versa, assume that $\mathcal{D}_x \equiv \mathcal{D}_x^* \equiv 0$. Let $\mathcal{S}^\#$ be the set of the following PDEs: $\mathcal{E}_z, \mathcal{E}_{\bar{z}}$, their derivatives up to order 2, the equation \mathcal{E}_0 and its derivative $(\mathcal{E}_0)_{;x}$. Then $\mathcal{S}^\#$ is involutive and hence solvable, and every solution K produces a cubic integral.

The whole set of solutions of $\mathcal{S}^\#$ can be constructed in the same way as it was done in the Proposition 3.1: We fix the initial value $K_{;x}(P)$ at some point $P \in S$, solve the ODE $\frac{\partial}{\partial x} K_{;x} = \mathcal{K}_{xx}^\#$ with the initial value $K_{;x}(P)$ along x -axis, then the ODE $\frac{\partial}{\partial y} K_{;x} = \mathcal{K}_{xy}^\#$ with the initial value $K_{;x}(x, 0)$ at $y = 0$, and finally express $K_{;y}$ using $K_{;y} = \mathcal{K}_{xy}^\#$. \square

5. INVARIANT EXPRESSIONS.

The complex calculus introduced and applied in previous sections relies on the choice of isothermic coordinates. In this paragraph we get rid of it: Lemma 5.1 gives an answer on the following two questions (for a fixed given metric):

- (1) Given an integral F cubic in momenta, how to calculate the tensor \hat{A} (the real part of the Birkhoff-Kolokoltsov form)?
- (2) Given a symmetric $(3, 0)$ -tensor A , how to understand whether it can be the real part of a holomorphic 2-codifferential.

If the coordinates we are working in are isothermal, it is easy to answer both questions using the definition. But if the coordinates are generic, the questions are not that trivial.

We would like to remark that the first question will be especially interesting in view of program for searching new metrics admitting cubic we integrals suggest in the conclusion Section 7.

We use the following notation. $x = x^1$ and $y = x^2$ are local coordinates, $g = g_{ij}dx^i dx^j$ is the metric tensor, $\lambda := \sqrt{\det(g_{ij})}$ is the volume density, so that $\omega_g = \lambda dx \wedge dy$ is the volume form, and J_i^j is the operator of the complex structure, ie., the operator of rotation by 90° in the tangent bundle (it is easy to construct it for an arbitrary metric).

Lemma 5.1. *i) Let $F = (F^{ijk})$ be a symmetric 3-vector (ie., $(3, 0)$ -tensor). Then it can be uniquely decomposed into the sum $F = \hat{A} + \hat{B}$ where $\hat{A} = (\hat{A}^{ijk})$ and $\hat{B} = (\hat{B}^{ijk})$ are also symmetric $(3, 0)$ -tensors, and $\hat{A} = \Re(A)$, $\hat{B} = \Re(B)$ for some sections A of the bundle $T^{[3,0]}S$ and B of the bundle $T^{[2,1]}S$. Moreover, for arbitrary 1-forms $\alpha_1, \alpha_2, \alpha_3$ one has*

$$\hat{A}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{4}(F(\alpha_1, \alpha_2, \alpha_3) - F(J\alpha_1, J\alpha_2, \alpha_3) - F(J\alpha_1, \alpha_2, J\alpha_3) - F(\alpha_1, J\alpha_2, J\alpha_3)) \quad (5.1)$$

$$\hat{B}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{4}(3F(\alpha_1, \alpha_2, \alpha_3) + F(J\alpha_1, J\alpha_2, \alpha_3) + F(J\alpha_1, \alpha_2, J\alpha_3) + F(\alpha_1, J\alpha_2, J\alpha_3)) \quad (5.2)$$

or in the index form

$$\hat{A}^{ijk} = \frac{1}{4}(F^{ijk} - F^{i'j'k} J_{i'}^i J_{j'}^j - F^{i'jk'} J_{i'}^i J_{k'}^k - F^{ij'k'} J_{j'}^j J_{k'}^k) \quad (5.3)$$

$$\hat{B}^{ijk} = \frac{1}{4}(3F^{ijk} + F^{i'j'k} J_{i'}^i J_{j'}^j + F^{i'jk'} J_{i'}^i J_{k'}^k + F^{ij'k'} J_{j'}^j J_{k'}^k) \quad (5.4)$$

ii) Let A be a section of the bundle $T^{[3,0]}S$ and $\hat{A} := \Re(A)$ its real part. Then the imaginary part $\Im(A)$ is given by

$$(\Im(A))^{ijk} = \frac{1}{3}(\hat{A}^{ljk} J_l^i + \hat{A}^{ilk} J_l^j + \hat{A}^{ijl} J_l^k). \quad (5.5)$$

Further, A is holomorphic if and only if the tensor $\hat{A} = (\hat{A}^{ijk})$ satisfies the equation $\nabla \hat{A}(J \cdot, J \cdot, J \cdot, J \cdot) = -\nabla \hat{A}(\cdot, \cdot, \cdot, \cdot)$ or in the index form

$$A^{(ijk;l)} = -A^{i'j'k';l'} J_{i'}^i J_{j'}^j J_{k'}^k J_{l'}^l, \quad (5.6)$$

where $(ijkl)$ means the symmetrisation in the induces.

iii) The principle equation $K_{;\bar{z}\bar{z}} = -\frac{i g^2}{2} A_{,z}$ can be written as

$$\frac{1}{2} K_{;kl} (\delta_i^k \delta_j^l + J^k_i J^l_j) + g_{k(i} \omega_{j)l} \hat{A}^{klm}_{;m} = 0 \quad (5.7)$$

where (ij) means the symmetrisation in indices.

Proof. *i)* By property (2) from § 2.4, the bundle of complex-valued symmetric 3-vectors is naturally isomorphic to the sum $T^{[3,0]}S \oplus T^{[2,1]}S \oplus T^{[1,2]}S \oplus T^{[0,3]}S$. The complex conjugation interchanges two inner and two outer summands. Therefore every real symmetric 3-vectors has the form $F = \Re(A + B)$ for uniquely defined sections A of $T^{[3,0]}S$ and B of $T^{[2,1]}S$.

The formulas for $\hat{A} = \Re(A)$ and $\hat{B} = \Re(B)$ are subject of linear algebra, therefore it is sufficient to check them for the Euclidean case (\mathbb{R}^2, g_{Euc}) and constant tensors $A = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$ and $B = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$. In this case $\hat{A} = \Re(A) = ((\frac{\partial}{\partial x})^3 - 3(\frac{\partial}{\partial x})^2 \frac{\partial}{\partial y})$, $\hat{B} = \Re(B) = \frac{\partial}{\partial x} ((\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2)$, and $J(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$, $J(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$, and an explicit verification follows.

ii) Consider the covariant derivative $\nabla \hat{A} = \Re(\nabla A)$. By (5) from § 2.4, in any complex coordinate z we obtain $\nabla A = A_{,z} dz + A_{,\bar{z}} d\bar{z}$. Again in the coordinate z , multiplication with $g^{-1} = \lambda^{-1} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ gives $g^{-1} \nabla A = A_{,z} \lambda^{-1} \frac{\partial}{\partial \bar{z}} + A_{,\bar{z}} \lambda^{-1} \frac{\partial}{\partial z}$. Thus $g^{-1} \nabla A$ is the sum of two components of the type $T^{[4,0]}S$ and $T^{[3,1]}S$, in which the $T^{[4,0]}S$ -component is “responsible” for Cauchy-Riemann term $A_{,\bar{z}}$. These two components are easily distinguished by the operator $J: C(J \cdot, J \cdot, J \cdot, J \cdot) = +C(\cdot, \cdot, \cdot, \cdot)$ for every section C of the bundle $T^{[4,0]}S$, and $C(J \cdot, J \cdot, J \cdot, J \cdot) = -C(\cdot, \cdot, \cdot, \cdot)$ for every section C of $T^{[3,1]}S$. Thus holomorphicity of A is equivalent to the equation $g^{-1} \nabla A(J \cdot, J \cdot, J \cdot, J \cdot) = -g^{-1} \nabla A(\cdot, \cdot, \cdot, \cdot)$.

Now we make use two easy observations. The first is that if C is a sum of two sections of the bundles $T^{[4,0]}S$ and $T^{[3,1]}S$ respectively, then the $T^{[4,0]}S$ -component of C vanishes if

and only if $\hat{C} := \Re C$ satisfies the equation $\hat{C}(J\cdot, J\cdot, J\cdot, J\cdot) = -\hat{C}(\cdot, \cdot, \cdot, \cdot)$. This fact can be deduced from the action of the complex conjugation on bundles $T^{[p,q]}S$, see (3) from §2.4. Another observation is that in the index form the symmetric tensor $\Re(g^{-1}\nabla A)$ is given by $A^{(ijk;l)} = A^{(ijk}{}_{;m}g^{l)m}$. This implies the formula (5.6).

Now consider the tensor $A_{;z}dz$. By the construction, this is the section of the bundle $T^{[3,0]}S \otimes \Omega^{[1,0]}S$. The latter bundle is isomorphic to $T^{[2,0]}S$, and the isomorphism is induced by the isomorphism $T^{[1,0]}S \otimes \Omega^{[1,0]}S \cong \mathbb{C}$. The latter isomorphism is simply the operation of contraction of indices. This operation commutes with complex conjugation, and therefore with the operator \Re of taking real part. It follows that the image of $\Re(A_{;z}dz)$ in $T^{[2,0]}S$ is given by the symmetric $(2,0)$ -tensor $\text{div}\hat{A}$ with components $(\hat{A}^{ijk}{}_{;k})$. It should be noticed that besides the usual symmetry $\hat{A}^{ijk}{}_{;k} = \hat{A}^{jik}{}_{;k}$ this tensor has another symmetric property, namely $\text{div}\hat{A}(J\cdot, J\cdot) = -\text{div}\hat{A}(\cdot, \cdot)$. Thus $\text{div}\hat{A}$ has two independent components, as it should be since $A_{;z}$ is complex-valued.

The last formula expressing $\Im(A)$ can be deduced in a similar way as above and is left to the reader as an easy exercise (and is not important for us).

iii) The real part of the principle equation (2.7) is

$$K_{;\bar{z}\bar{z}} + K_{;zz} = \frac{ig^2}{2}(\bar{A}_{;\bar{z}} - A_{;z}). \quad (5.8)$$

Since the principle equation is the $\Omega^{[0,2]}$ -component of (5.8), both equations are equivalent. Since $T(Jv, Jw) = T(v, w)$ for every section T of the bundle $\Omega^{[1,1]}S$ and $T(Jv, Jw) = -T(v, w)$ for sections of $\Omega^{[2,0]}S \oplus \Omega^{[0,2]}S$, the tensor $K_{;\bar{z}\bar{z}} + K_{;zz}$ has components

$$\frac{1}{2}K_{;kl}(\delta_i^k\delta_j^l + J^k{}_i J^l{}_j).$$

Rewriting the right hand side of (5.8) we start with the equality $\frac{i}{2}(\bar{A}_{;\bar{z}} - A_{;z}) = \Im(A_{;z})$. Since A is holomorphic and $A_{;\bar{z}} = 0$, the tensor $A_{;z}$ equals $A^{ijk}{}_{;k}$. Using the fact that the isomorphism (2.2) is simply the contraction of indices, we obtain the formula $g_{ik}g_{jl}\Im(A^{klm}{}_{;m})$. Further, since $A_{;z}$ has type $T^{[2,0]}S$, $i\cdot A_{;z} = J\cdot A_{;z}$ and hence $\Im(A_{;z}) = -\Re(J\cdot A_{;z})$. Using the equality $g_{ij}J^j{}_k = \omega_{ik}$ we obtain the desired form (5.7) of the equation. \square

6. PSEUDO-RIEMANNIAN CASE.

In this section we show that the results and formulas obtained in the case of Riemannian metrics remain valid in the pseudo-Riemannian case after an appropriate modifications which we describe.

6.1. Null-coordinates. These are the counterpart of isothermic coordinates in the pseudo-Riemannian case.

Definition 6.1. Let g be a pseudo-Riemannian metric on a surface. A vector (field) v satisfying $g(v, v) = 0$ is called a *null-vector (field)*. *Null-coordinates*² are such coordinates in which g has the anti-diagonal form $g = \lambda(x, y)dxdy$.

Lemma 6.1 (Folklore). *Let g be a pseudo-Riemannian metric on a surface S . Then at each point on S there exist local coordinates $x = x^1, y = x^2$ in which g has the anti-diagonal form $g = \lambda(x, y)dxdy$*

²From the mathematical point of view, *isotropic coordinate system* would be probably a better notion. However, this terminology is already established in the physical literature, see e.g. [Cu-La], and moreover, the notion *isotropic coordinates* itself has another meaning, see e.g. [Cro].

Proof. At each point p on S the metric g has exactly two isotropic directions, ie., pointwisely linearly independent vectors fields v, w with $g(v, v) = g(w, w) = 0$. Integrating them we obtain a 2-web of curves on S . The local parameters functions for this 2-web are the desired coordinates $x = x^1, y = x^2$. \square

Let us locally fix the order and the orientations of null-directions. Then an null-coordinate system is defined uniquely up to oriented reparametrisations $x^i = f^i(x^i)$.

6.2. Calculus in null-coordinates. As in the Riemannian case, formulas and calculus become much simpler if we use null-coordinates.

In analogy with the complex calculus, define *real* bundles $T^{[1,0]}S, T^{[0,1]}S, \Omega^{[1,0]}S, \Omega^{[0,1]}S$, generated respectively by vectors $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ and the forms dx^1, dx^2 . The definition of these bundles depends only on the local combinatorial data, whereas the decompositions $TS = T^{[1,0]}S \oplus T^{[0,1]}S$ and $T^*S = \Omega^{[1,0]}S \oplus \Omega^{[0,1]}S$ are defined solely by the metric g . Further, for $p, q \geq 0$ set

$$T^{[p,q]}S := (T^{[1,0]}S)^{\otimes p} \otimes (T^{[0,1]}S)^{\otimes q} \quad \text{and} \quad \Omega^{[p,q]}S := (\Omega^{[1,0]}S)^{\otimes p} \otimes (\Omega^{[0,1]}S)^{\otimes q}.$$

Then

$$T^{[p,q]}S \otimes T^{[p',q']}S = T^{[p+p',q+q']}S \quad (6.1)$$

for non-negative p, p', q, q' . We extend the definition of the bundles $T^{[p,q]}S, \Omega^{[p,q]}S$ for all integers p, q by means of the formula (6.1) and the relation

$$\Omega^{[p,q]}S = T^{[-p,-q]}S. \quad (6.2)$$

Every section of the bundle $T^{[p,q]}S$ or $\Omega^{[p,q]}S$ is given by its single component/coefficient which is a real function.

As in the case of complex calculus, we imbed the bundles $T^{[p,q]}S, \Omega^{[p,q]}S$ with $p, q \geq 0$ in the bundles of *symmetric* $(k, 0)$ - and $(0, k)$ tensors. The imbedding is done using symmetrisation operator. This gives us natural decompositions $\text{Sym}^k TS = \bigoplus_{p+q=k}^k T^{[p,q]}S$ and $\text{Sym}^k T^*S = \bigoplus_{p+q=k}^k \Omega^{[p,q]}S$.

The metric g is a section of the bundle $\Omega^{[1,1]}S$. We use multiplication with powers of g to define natural isomorphisms $T^{[p,q]}S \xrightarrow{\cong} T^{[p+k,q+k]}S$ and $\Omega^{[p,q]}S \xrightarrow{\cong} \Omega^{[p+k,q+k]}S$.

Further, define the operators $\nabla^{[1,0]} : \Omega^{[p,q]}S \rightarrow \Omega^{[p+1,q]}S$ and $\nabla^{[0,1]} : \Omega^{[p,q]}S \rightarrow \Omega^{[p,q+1]}S$ as the corresponding homogeneous components of the operator ∇ . Thus for any section $F = f(x, y)dx^p dy^q$ of the bundle $\Omega^{[p,q]}S$ with $p, q \geq 0$ we obtain

$$\nabla^{[1,0]}F = f_{;x}dx^{p+1}dy^q, \quad \nabla^{[0,1]}F = f_{;y}dx^p dy^{q+1},$$

and $\nabla^{[1,0]}F + \nabla^{[0,1]}F = \text{Sym}(\nabla F)$ where Sym denotes the symmetrisation operator. In the case of a section $A = a(x, y)(\partial_x)^p(\partial_y)^q$ of the bundle $T^{[p,q]}S$ with $p, q \geq 0$ we obtain

$$\nabla^{[1,0]}A = a_{;x}(\partial_x)^{p+1}(\partial_y)^q \quad \nabla^{[0,1]}A = a_{;y}(\partial_x)^p(\partial_y)^{q+1},$$

and $\nabla^{[1,0]}A + \nabla^{[0,1]}A = \text{Tr}(\nabla A) = \text{div}(A)$ where Tr denotes the contraction of indices. The operators $\nabla^{[1,0]}$ and $\nabla^{[0,1]}$ do not commute, and the commutator formula is similar to the complex case (2.3):

$$\begin{aligned} (\nabla^{(1,0)}\nabla^{(0,1)} - \nabla^{(0,1)}\nabla^{(1,0)})f dx^p dy^q &= \frac{q-p}{2} \cdot R \cdot g \cdot f dx^p dy^q \\ (\nabla^{(1,0)}\nabla^{(0,1)} - \nabla^{(0,1)}\nabla^{(1,0)})f(\partial_x)^p(\partial_y)^q &= \frac{p-q}{2} \cdot R \cdot g \cdot f(\partial_x)^p(\partial_y)^q \end{aligned} \quad (6.3)$$

Let $F(x, y; p_x, p_y)$ be a function on T^*S cubic in momenta. Then it has a form $F = A_1 p_x^3 - A_2 p_y^3 + (B_1 p_x + B_2 p_y) p_x p_y$ for uniquely defined sections A_1 of $T^{[3,0]}S$, A_2 of $T^{[0,3]}S$, B_1 of $T^{[2,1]}S$, and B_2 of $T^{[2,1]}S$. The Hamiltonian function corresponding to the metric has the form $H = \frac{p_x p_y}{2\lambda(x,y)}$, and the equation of integrability of F reads

$$\begin{aligned} \{F, H\} = & \frac{1}{2\lambda^2} \left(p_x^4 \lambda \cdot (A_1)_{;y} - p_y^4 \lambda \cdot (A_2)_{;x} + \right. \\ & p_x^3 p_y \cdot (B_1 \cdot \lambda_{;y} + 3A_1 \cdot \lambda_{;x} + \lambda \cdot (B_1)_{;y} + \lambda \cdot (A_1)_{;x}) + \\ & p_x p_y^3 \cdot (B_2 \cdot \lambda_{;x} - 3A_2 \cdot \lambda_{;y} + \lambda \cdot (B_2)_{;x} - \lambda \cdot (A_2)_{;y}) + \\ & \left. p_x^2 p_y^2 \cdot (2B_1 \cdot \lambda_{;x} + \lambda \cdot (B_1)_{;x} + 2B_2 \cdot \lambda_{;y} + \lambda \cdot (B_2)_{;y}) \right) \end{aligned} \quad (6.4)$$

Lemma 6.2. *Let $g = \lambda dx dy$ be a pseudo-Riemannian metric on a surface S in null-coordinates.*

(a) *A function $a(x, y)$ is independent of the variable y if and only if for some/every $k \geq 0$ the tensor $A := a \partial_x^k$ satisfies the equation $A_{;y} = 0$.*

(b) *Let F be an integral of g cubic in momenta. Then its components A_1 and A_2 depend only on one corresponding variable.*

Moreover, if A_1, A_2 are non-zero at a given point on S , then there exists a null-coordinate system (x, y) , unique up to exchange of the coordinates, in which components of A_i are identically 1, i.e., $A_1 \equiv \partial_x^3$ and $A_2 \equiv \partial_y^3$.

Here $A_{;y}$ means the covariant derivative $\nabla^{[0,1]}A$.

Remark 6.1. *Similar statement holds for polynomial integrals of other degree, see [Bo-Ma-Pu1, §3.3] for the case of quadratic integrals.*

Proof. If (x', y') is any null-coordinate system with $A_1 = a_1(x') \partial_{x'}^3$ and $A_2 = a_2(y') \partial_{y'}^3$, then the desired coordinates are defined by $x = \int \frac{dx'}{\sqrt[3]{a_1(x')}}$ and $y = \int \frac{dy'}{\sqrt[3]{a_2(y')}}$. \square

Definition 6.2. For any function F on the cotangent bundle T^*S cubic in momenta we call the components A_1 and A_2 the *Birkhoff-Kolokoltsov 3-codifferentials* associated with F and pseudo-Riemannian metric g . A_1 and/or A_2 is said to be *quasi-holomorphic*³ if $A_i = a_i(x^i) \partial_{x^i}^3$ with $a_i(x^i)$ depending only on one variable.

The null-coordinate system (x, y) in which $A_1 \equiv \partial_x^3$ and $A_2 \equiv \partial_y^3$ is called *adapted* to A_1, A_2 or to F .

Proposition 6.1. *Let $g = \lambda dx dy$ be a pseudo-Riemannian metric on a surface S in null-coordinates and F a function cubic in momenta. Then F is an integral for g if and only if*

- (A) *its components A_1, A_2 are quasi-holomorphic;*
- (B) *there exists a function $K(x, y)$ such that the components B_1, B_2 are related as $B_1 = -\lambda^{-2} K_{;y}$, $B_2 = \lambda^{-2} K_{;x}$;*
- (K) *the function K satisfies the equations*

$$K_{;xx} = g^2 (A_2)_{;y} \quad K_{;yy} = g^2 (A_1)_{;x}.$$

³In view of pseudo-Riemannian metrics, pseudo-holomorphic would be probably a better notion. However, this terminology is already reserved, see [Gro].

Proof. Five homogeneous components of bi-degree $T^{[i,j]}S$ ($i + j = 4$) of the Poisson bracket (6.4) are five equations on A_i, B_i and λ . The function F is a cubic integral for $g = \lambda dx dy$ iff all five equations are satisfied. The first two are equations (A). The last one can be written as $d(\lambda^2 B_1 dx + \lambda^2 B_2 dy) = 0$ and hence is equivalent to the property (B). Finally, substituting expressions (B) into the remaining two equations (second and third lines in (6.4)) we obtain equations (K). \square

6.3. Calculation of equations in pseudo-Riemannian case. Here we use the trick and translate all the results of §2.3 and Section 4 almost literally in the new situation. Doing so, we must follow the next rules.

- the complex coordinate z is replaced by the null-coordinate x , the coordinate \bar{z} by y ;
- the holomorphic 3-codifferential A is replaced by the quasi-holomorphic 3-codifferential A_1, \bar{A} by A_2 ,
- the imaginary unit i is dropped (replaced by 1), the operator J^i_j of rotation by 90° is replaced by the identity Id operator;
- the complex conjugation $a \mapsto \bar{a}$ is replaced by the *null-involution* $I : (x, y) \mapsto (y, x)$; the operator $\Im(A) = \frac{1}{2i}(a - \bar{a})$ transforms into the operator $P_- : a \mapsto \frac{1}{2}(a - I(a))$, this is the operator of the projection on the eigenspace $\mathbb{E}_I(-1)$ of the operator I corresponding to the eigenvalue -1 ; similarly, the operator $\Re(A) = \frac{1}{2}(a + \bar{a})$ transforms into the operator $P_+ : a \mapsto \frac{1}{2}(a + I(a))$ corresponding to the eigenvalue $+1$;
- the Poisson bracket $\{f, g\} = \frac{1}{2ig}(f_{;z}g_{;\bar{z}} - f_{;\bar{z}}g_{;z}) = g^{-1}\Im(f_{;z}g_{;\bar{z}})$ is now given by the new formula $\{f, g\} := g^{-1}P_-(f_{;x}g_{;y}) = \frac{1}{2g}(f_{;x}g_{;y} - f_{;y}g_{;x})$;
- if it is possible, every equation must be written in complex form and then its conjugate must included to the system; exceptions are the equations which are real “in their nature”, such as $\mathcal{E}_0, \dots, \mathcal{E}_3; \mathcal{E}_1^*, \dots, \mathcal{E}_3^*, \mathcal{G}_0, \dots, \mathcal{G}_3$; then after applying the above rules every pair of complex conjugate equations transforms into a pair of equations $\mathcal{E}, I(\mathcal{E})$ where I is the null-involution above

For example, the equation $K_{;\bar{z}\bar{z}} = A_{;z}$ is complemented by its conjugate $K_{;zz} = A_{;\bar{z}}$, so that counterpart of $K_{;\bar{z}\bar{z}} = A_{;z}$ in the pseudo-Riemannian case is the pair of the equations $K_{;xx} = A_{;y}$ and $K_{;yy} = A_{;x}$. In particular, we have two real equations in both Riemannian and pseudo-Riemannian cases.

Proposition 6.2. (a) *After changes made by the rules described above, Theorems 1.1 and 1.2 remain valid in the pseudo-Riemannian case.*

(b) *Let g be a pseudo-Riemannian metric with vanishing φ_2, φ_2^* and non-vanishing φ_1 . Then g admits a nontrivial linear integral.*

Proof. (a) The proof of Theorems 1.1 and 1.2 is done by means of formal calculus in which the condition of positivity of the metric was never used.

(b) Considering in [Ei, pp. 323-325] Riemannian metrics g on surfaces with non-constant curvature R , Eisenhart introduces the following coordinate system: One of the coordinates is the curvature R itself and the other is orthogonal to the first one. In other words, $g = g_{11}dx^2 + g_{22}dy^2$ and $R(x, y) = x$. The necessary and sufficient condition for the existence of such coordinates is that ∇R is not orthogonal to itself. In the Riemannian case this simply means the non-vanishing of ∇R and thus the non-constancy of R , in the pseudo-Riemannian case this is the condition $\varphi_1 \neq 0$. The rest of Eisenhart’s proof works also in pseudo-Riemannian. \square

6.4. Pseudo-Riemannian metrics with $|\nabla R|_g^2 = 0$: local classification and nonexistence of cubic integrals. The goal of this paragraph is to prove

Proposition 6.3. *If a pseudo-Riemannian metric g with $|\nabla R|_g^2 \equiv 0$ admits a non-trivial cubic integral, then R is constant.*

We will need the following

Lemma 6.3. *Let g be a pseudo-Riemannian metrics with $\nabla R \neq 0$ and $|\nabla R|_g^2 = 0$. Then in generic null-coordinates it has the form*

$$g = \frac{f_2(x)f_1(y)_y dx dy}{(f_1(y) + f(x))^2} \quad (6.5)$$

with some functions $f_1(y), f(x), f_2(x)$, and then the curvature is given by

$$R = \frac{2f(x)_x}{f_2(x)}.$$

Moreover, there exist null-coordinates in which g has the normal form

$$g = \frac{dx dy}{(y + f(x))^2} \quad \text{with} \quad R = 2f(x)_x. \quad (6.6)$$

where $f(x)$ is some function.

Such a metric g admits no Killing vector field, ie., no linear integral L .

Remark. The generic formula is given up to exchange of null-coordinates (x, y) . In (6.5) and in the proof subscript indices $f_1(y)_y, u_{xyy}, \dots$ denote usual (partial) and *not covariant* derivatives of functions $f_1(y), u(x, y), \dots$

Proof. Choose some null-coordinates (x, y) . The condition $|\nabla R|_g^2 = 0$ means that the gradient ∇R is a null-vector. Hence R depends only on one of null-coordinates, say on x , and $R_y = 0$. On the other hand, $R_x \neq 0$ by the hypotheses of the lemma. Inverting the coordinate x if needed, we may assume that $g = e^{u(x,y)} dx dy$ for some function $u(x, y)$. Then $R = e^{-u} u_{xy}$ and the equation $|\nabla R|_g^2 = 0$ reads $(e^{-u} u_{xy})_y = 0$, or $u_{xyy} - u_{xy} u_y = 0$. Substitution $u_y = v$ yields $v_{xy} - v v_x = 0$, which is equivalent to $(2v_y - v^2)_x = 0$. Hence $2v_y = v^2 + \psi(y)$ for some function $\psi(y)$. This is a Riccati ODE, whose generic solution can be found as follows. Let $v_1(y)$ be a fixed special solution of the equation $2v_y = v^2 + \psi(y)$, so that $\psi(y) = 2(v_1(y))_y - v_1(y)^2$. Since $\psi(y)$ was arbitrary, we may assume that $v_1(y)$ is an arbitrary function and $\psi(y)$ is given by the above formula. Now make the substitution $v(x, y) = v_1(y) + \frac{2}{w(x,y)}$. Then the equation $2v_y = v^2 + \psi(y)$ transforms into $w_y + v_1 w + 1 = 0$. Since $v_1(y)$ is an arbitrary function we may assume it has the form $v_1(y) = \frac{f_1(y)_{yy}}{f_1(y)_y}$ with an arbitrary function $f_1(y)$. Then the equation $w_y + v_1 w + 1 = 0$ reads $f_1(y)_y w_y + f_1(y)_{yy} w + f_1(y)_y = 0$, which means $(f_1(y)_y w(x, y) + f_1(y))_y = 0$. Consequently, $w(x, y) = -\frac{f_1(y) + f(x)}{f_1(y)_y}$ with an arbitrary function $f(x)$, and

$$v(x, y) = \frac{f_1(y)_{yy}}{f_1(y)_y} - 2 \frac{f_1(y)_y}{f_1(y) + f(x)}.$$

Now integration yields $u(x, y) = \log(f_2(x)) + \int v(x, y) dy$ which gives the desired form of the metric $e^{u(x,y)} dx dy$.

The functions $f_1(y), f_2(x)$ can be easily eliminated using the substitution $y' = f_1(y)$, $x' = \int f_2(x) dx$.

Now assume that a metric $g = \lambda(x, y)dx dy$ with $R_y = 0$ admits a linear integral L . This is equivalent to the existence of Killing vector field ξ . Since the curvature R is constant on lines $x = c$, ξ must have the form $\xi = \alpha(x, y)\partial_y$. The Lie derivation gives

$$\mathcal{L}_\xi(\lambda dx dy) = \alpha \partial_y \lambda dx dy + \lambda \partial_x \alpha dx dx + \lambda \partial_y \alpha dx dy.$$

In particular, $\partial_x \alpha$ must vanish. Then α depends only on y , and after the appropriate reparametrisation $\xi = \partial_y$. But then λ must be independent of x which would imply the vanishing of R . \square

Proof of Proposition 6.3. We use the following notation: x, y are null-coordinates in which the metric has the form (6.6), $f = f(x)$, $f_1(x) = f_1$, $f_2(x) = f_2$ are functions of the variable x , $f'_1, f''_1, f'''_1, f''_2, f'''_2$ their derivatives. Further, we set $A_1 = a_1(x)\partial_x^3$, $A_2 = a_2(y)\partial_y^3$, and then a'_1, a''_1 denote the derivatives (in y in the case a_2). The usual (non-covariant!) partial derivatives are denoted by $\partial_{xxy}^3 K$ and so on. Besides we introduce the function $Y(x, y) := y + f(x)$. Clearly, it satisfies the relations $\partial_x Y = f'$ and $\partial_y Y = 1$. Finally, c_1, c_2, \dots will be constants.

In the case $g = \frac{dx dy}{Y^2}$ with $Y = y + f(x)$ the only non-zero Christoffel symbols are $\Gamma_{11}^1 = -\frac{2f'}{Y}$ and $\Gamma_{22}^2 = -\frac{2}{Y}$. This gives us the expansions

$$K_{;yy} = \partial_{yy}^2 K + \frac{2}{Y} \partial_y K \quad \text{and} \quad g^2 A_{1;x} = \frac{Y a'_1 - 6f' a_1}{Y^5}.$$

Thus $K_{;yy} - g^2 A_{1;x}$ appears to be an ODE

$$\partial_{yy}^2 K + \frac{2}{y+f} \partial_y K - \frac{(y+f)a'_1 - 6f'a_1}{(y+f)^5} = 0$$

on K with respect to y with rational coefficients in which x is a parameter. The direct integration gives

$$K(x, y) = f_1(x) + \frac{f_2(x)}{Y} + \frac{Y a'_1 - 2f' a_1}{2Y^3}, \quad (6.7)$$

and hence

$$\partial_x K(x, y) = f'_1 + \frac{f'_2}{Y} - \frac{f' f_2}{Y^2} + \frac{a''_1}{2Y^2} - \frac{f' a'_1}{Y^3} - \frac{f' a'_1 + f'' a_1}{Y^3} + \frac{3f'^2 a_1}{Y^4} \quad (6.8)$$

In the same way we obtain

$$K_{;xx} = (\partial_x + \frac{2f'}{Y}) \partial_x K \quad \text{and} \quad g^2 A_{2;y} = \frac{Y a'_2 - 6a_2}{Y^5}.$$

Now we make the following observation: Substituting of (6.7) in $Y^5(\partial_{xx}^2 K + \frac{2f'}{Y} \partial_x K)$ and using $\partial_x Y = f'$, $\partial_y Y = 1$ we obtain a polynomial (say $P(Y)$) of degree 5 in Y whose coefficients are smooth functions in x , such that $a_2(y)$ satisfy the equation $Y \partial_y a_2 - 6a_2 = P(Y)$. For any fixed x this is a polynomial ODE on a_2 . Integrating it, we obtain that (for any fixed x !) is given by

$$a_2(y) = Y^6 \cdot (f_3(x) + \int Y^{-7} P(Y) dY). \quad (6.9)$$

In view of this formula, the problem of compatibility of the 3-codifferential $A = (A_1, A_2)$ with the metric $g = \frac{dx dy}{(y+f(x))^2}$ is equivalent to finding the functions $f(x), f_1(x), f_2(x), f_3(x), a(x)$ whose substitution in (6.9) gives a function independent of x .

Further, if we write $P(Y) = \sum_{i=0}^5 p_i Y^i$, then the integration of (6.9) gives a polynomial expression for a_2

$$a_2 = f_3 Y^6 - \sum_{i=0}^5 \frac{p_i}{6-i} Y^i$$

with coefficients depending only on x . After expanding $Y = y + f(x)$ we still obtain an expression for a_2 which is a polynomial $Q(y) = \sum_{j=0}^6 q_j(x) y^j$ in y of degree 6 with coefficients $q_j(x)$ depending only on x . However, since a_2 is independent of y , all these coefficients must

be constant. In this way we obtain 6 equations $q_i(x) = c_i$ (ODE in general) on the above functions $f(x), f_1(x), f_2(x), f_3(x)$ and $a_1(x)$. The expressions $q_i(x)$ can be calculated (or checked) using the formulas above, and we simply provide the final results.

We are going to solve these equations using the prolongation (few times) and projection (mostly) method, resolving subsequently equations $q_6(x) = c_6, q_5(x) = c_5, \dots$ and substituting the results into the next equation. The obtained new ODEs will be denoted by E_6, E_5, \dots

The first equation is simply $f_3(x) = c_6$, so f_3 is a constant. The next equation $q_5(x) = c_5$ (up to constant factor) reads as $c_5 + 6c_6f - f_1'' = 0$ or

$$f_1'' = c_5 + 6c_6f. \quad (6.10)$$

Substituting the latter expression $q_4(x) = c_4$ we obtain the next ODE E_4 which reads $c_4 - 15c_6f^2 - f_1'f' - \frac{1}{2}f_2'' - 5fc_5$. Resolving it we obtain

$$f_2'' = -2f_1'f' - 10fc_5 - 30c_6f^2 + 2c_4. \quad (6.11)$$

Next one substitution gives the equation E_3 which can be resolved as follows:

$$a_1''' = 6c_3 + 120c_6f^3 + 2f_2f'' + 60f^2c_5 - 24fc_4. \quad (6.12)$$

Doing the same with $q_2(x) = c_2$, we obtain

$$a_1'' = (2f')^{-1}(-2c_2 - 3a_1'f'' - a_1f''' + 12fc_3 + 60c_6f^4 + 40f^3c_5 - 24f^2c_4). \quad (6.13)$$

The result of the same procedure for $q_1(x) = c_1, q_0(x) = c_0$ is

$$a_1' = (10f'^2)^{-1}(30f^2c_3 + 60c_6f^5 - 14a_1f''f' - 40f^3c_4 + 50f^4c_5 + 5c_1 - 10fc_2), \quad (6.14)$$

$$a_1 = (2f'^3)^{-1}(2f^4c_4 - 2f^3c_3 + c_0 - 2f^5c_5 - 2c_6f^6 - f(x)c_1 + f^2c_2). \quad (6.15)$$

Recall that we have substituted in each successive equation the results of preceding calculations. This was clearly the projection procedure, and the next one is the prolongation one. Let us denote the expressions in (6.12–6.15) by $\mathcal{A}_3, \dots, \mathcal{A}_0$, so that the formulas read $a_1^{(i)} = \mathcal{A}_i$. We compute solely the consistency equation $\mathcal{A}_1 - \partial_x \mathcal{A}_0$. The result is

$$\mathcal{A}_1 - \partial_x \mathcal{A}_0 = \frac{f''}{10f'^4}(16c_6f^6 + 16f^5c_5 - 16f^4c_4 + 16f^3c_3 - 8f^2c_2 + 8fc_1 - 8c_0). \quad (6.16)$$

Since this consistency condition must be satisfied identically, we conclude that there are two possibilities: either f'' vanishes identically, or all constants c_0, \dots, c_6 must be zero. By (6.6), $f'' \equiv 0$ means that the curvature R is constant.

We exclude this possibility and consider the alternative case $c_0 = \dots = c_6 = 0$. Then from (6.15) we see that a_1 vanishes identically. The equation (6.12) reads now $f_2f'' = 0$, and so f_2 also vanishes. Finally, from (6.11) we obtain $f_1'f' = 0$, which means that $f_1(x)$ is constant.

Summing up, we conclude from (6.7) that the the function $K(x, y)$ is constant. Finally, let us observe that the equation $(y + f(x))\partial_y a_2(y) = 6a_2(y)$ admit no non-trivial solution depending only on y . \square

7. CONCLUSION

We presented an algorithm that, given a pair (g, A) , where g is a two-dimensional metric, and A is a holomorphic 3-codifferential, answers the question whether there exists a cubic integral whose Birkhoff-Kolokoltsov 3-codifferential coincides with A . Moreover, in the most interesting cases covered by Theorems 1.1, 1.2, it provides precise formulas for the integral. The algorithm works in an arbitrary coordinate system: we need to calculate certain expressions given by precise algebraic formulas including the components of g and A and their derivatives, and compare them with zero. It is easy to implement the algorithm on modern computer algebra packages, say Maple[®] or Mathematica[®].

Our results suggest the following program for search for new natural Hamiltonian system admitting integrals polynomial in momenta integrals of degree ≤ 3 .

Let us take a metric on a surface whose geodesic flow admits a cubic integral. For example, we can take a metric of constant curvature, or a metric admitting a Killing vector field, or a metric coming from one of the known natural Hamiltonian system via Maupertuis principle.

Let $K := |\vec{p}|_g$ be the kinetic energy corresponding to such metric and F_3 be the integral. Let us now look for a function $F_1 : T^*S \rightarrow \mathbb{R}$ linear in momenta, and for a function $V : S \rightarrow \mathbb{R}$ such that $F_3 + F_1$ in an integral for $K + V$, i.e., $\{F_3 + F_1, K + V\} = 0$, where K is the kinetic energy corresponding to g . By Maupertuis principle, the later condition is equivalent to the statement that for every constant h the functions $H_h := \frac{1}{(h-V)}K_g$ and $F_h := F_3 + H_h \cdot F_1$ commute: $\{H_h, F_h\} = 0$.

Clearly, the Birkhoff-Kolokoltsov 3-codifferential of the pair H_h, F_h does not depend on the parameter h and coincides with the Birkhoff-Kolokoltsov 3-codifferential of the pair K, F_3 , i.e., is known since we know K and F_3 . Thus, the conditions $\mathcal{G}_2 = 0$, $\mathcal{G}_3 = 0$ (or $\mathcal{D} = 0$, $\mathcal{G}_2^* = 0$, $\mathcal{G}_3^* = 0$ in the case covered by Theorem 1.2) can be viewed as quasilinear PDEs on the coefficients of F_1 and on V , i.e., on three unknown functions. Since the conditions are fulfilled for every h , the system of PDEs for V and F_1 is overdetermined. Easy analysis shows that it is of finite type and there exists an algorithmic (though highly computational) way to find a solution.

In other words, we suggest to look for new systems by adding potential to the known integrable geodesic flows, and the results of our paper ensures that one can do it algorithmically.

Note that all known natural Hamiltonian systems either come from physics (e.g. Goryachev-Chaplygin), or were obtained by the naive version of the above procedure (for example those obtained in [Du-Ma, Se, Ve-Ts, Ye]).

As the most promising metrics g to start the above program, we consider the metrics admitting linear integrals. Recall that such metric are completely described. Local description is due at least to [Da, §§590–593], global (= when the manifold is closed) can be found for example in [Bo-Ma-Fo]. Note that the starting point for the systems from [Du-Ma, Se, Ve-Ts, Ye] were certain metrics admitting linear integrals. This will be the next direction of our research; we are quite positive that it is possible to describe all natural system admitting integrals polynomial in momenta of degree ≤ 3 such that the corresponding metric admits a Killing vector field, and hope to find new examples.

Another perspective application is to try to prove/disprove the following conjecture from [Bo-Ko-Fo]: *if a real-analytic metric on the 2-torus admits a cubic integral, then it admits a linear integral*, or its weaker form: *if a real-analytic natural Hamiltonian system on the 2-torus admits an integral of degree three in momenta, then it admits a linear integral*. Indeed,

as we explained in Corollary 2.1(ii), the space of holomorphic 3-codifferential on the torus is a 2-dimensional linear space. More precisely, there exists a global (periodic) coordinate system (x, y) on the torus such that the metrics has the form $\lambda(x, y)(dx^2 + dy^2)$; in this coordinates system the form A is $(a + bi)\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z}$, where $a, b \in \mathbb{R}$. Then, one can view the equations $\mathcal{G}_2 = 0$, $\mathcal{G}_3 = 0$ on the torus as two PDE on the coefficient λ depending on two parameters a, b .

One should also mention that in paragraph 2.4 we invented a calculus adapted to our problem. The calculations are much easier in this calculus (for example, in the recent papers [Kr] and [Br-Du-Ea], which deal with a priori easier quadratic integrals, most calculations were done using computer algebra programs; in our paper, everything is done “by hand”). We expect that the same calculus will effectively work in the case of integrals of higher degree.

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